ME301 THEORY OF MACHINES 1- MECHANISMS
KINEMATIC ANALYSIS OF MECHANISMS

We shall consider planar mechanisms only.

In this chapter we shall assume that we know the dimensions of all the links. If the mechanism has $F$ degrees of freedom, we shall assume that we know the value of $F$ number of parameters.

Our aim is:

1. Determine the position of all the links in the mechanism
2. Determine the paths of points on these links
3. To determine velocity and acceleration characteristics of all the links or points on these links.
**Position:** Location of a rigid body (link) or a particle (point) in a rigid body with respect to a given reference frame.

**Path:** Locus of successive positions of a particle (point) on a rigid body (link).

**Displacement:** Change in position of a rigid body (link) or a particle (point) with respect to a reference frame. It is a vector quantity whose magnitude is called *distance* (measured in mm or m).

**Velocity:** The rate of change of position of a particle or a rigid body. It is the time rate of change of displacement. It is a vector quantity whose magnitude is called speed (mm/sec = mms$^{-1}$ or m/sec = ms$^{-1}$).

**Acceleration:** Time rate of change of velocity. It is a vector quantity whose magnitude is measured in mm/sec$^2$ = mms$^{-2}$ or m/sec$^2$ = ms$^{-2}$.
Kinematics of a Particle

\[\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}\]

\[\vec{r} = \hat{i} x + \hat{j} y\]

\[r = \sqrt{x^2 + y^2}\]

\[\theta = \tan^{-1}\left(\frac{y}{x}\right)\]

\[\vec{r} = r \angle \theta\]
Complex Numbers

Real Numbers: are used to represent the magnitude of a quantity.

When a real number is operated by (-1) operator, that number is rotated by 180°.
(i) operator: when this operator operates on a real number, that number is to be rotated by $90^0$ CCW.

If we operate on a real number by $i$ twice: $(i^2)b$, the real number must rotate twice by $90^0=180^0$. Since 1800 rotation is defined by (-1) operator:

$$i^2 = -1$$

$Ib$ is the “imaginary number”, shows the real number $b$ rotated $90^0$ CCW.

$a+ib$ is the “complex number”. It shows the location of a point $P$ in the complex plane (Cauchy plane or Gauss-Argand Diagram)

$r$ is the \textit{modulus}, $\theta$ is the \textit{argument} of the complex number.

\[ r = \sqrt{a^2 + b^2} \]

\[ \theta = \tan^{-1}\left(\frac{b}{a}\right) \]
1. Two Complex numbers can be equal only if their real and imaginary parts are equal.

2. Complex numbers add vectorially (Paralelogram Law of addition)

   The sum of two complex numbers is determined by adding real and imaginary parts separately. If \( c_1 = a_1 + ib_1 \) and \( c_2 = a_2 + ib_2 \) then
   \[
   z = c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2)
   \]

3. Multiplication and division follows the rules of ordinary algebra with the additional relation \( i^2 = -1 \).
\( c = a + ib \) : orthogonal representation

\[ c = r(\cos \theta + i \sin \theta) \]

Euler's equation: \( e^{i\theta} = \cos \theta + i \sin \theta \),

\[ c = r e^{i\theta} \] : Exponential form

Multiplication of a complex number by a constant, \( k \)

\[ u = e^{i\theta} \] a unit vector making an angle \( \theta \) wr to real axis

Multiplication of a complex number by \( e^{i\phi} \).
Conjugate of a complex number:

\[ \bar{c} = a - ib \]

\[ c = a + ib \]

\[ \bar{c} = \text{re}^{i\theta} \]

\[ c = \text{re}^{-i\theta} \]

\[ r^2 = c\bar{c} = (a+ib)(a-ib) = a^2 + b^2 \]

Real part of \( c \):

\[ \text{Re} [c] = \frac{1}{2} [(a+ib) + (a-ib)] = \frac{1}{2}(c + \bar{c}) \]

Imaginary part of \( c \):

\[ \text{Im} [c] = \frac{1}{2} [(a+ib) - (a-ib)] = \frac{1}{2}[(c - \bar{c})] \]

Differentiation of complex numbers also follows the rules of ordinary calculus.
\[ z = r \cos \theta + i r \sin \theta \]
\[ z = r (\cos \theta + i \sin \theta) \]

\[ e^{\pm i \theta} = \cos \theta \pm i \sin \theta \]

\[ z = re^{i \theta} \]
**Kinematics of Rigid Body in Plane**

The assumption of rigidity results with the following three important conclusions:

1. The plane motion of a rigid body is completely described by the motion of any two points within the rigid body or by a point and the angle a line on the rigid plane makes wr to a reference.

2. Rigidity ensures that the particles lying on a straight line have equal velocity components in the direction of this line, since the distance between any two points along this line remains constant.

3. We are concerned with the kinematics of the rigid bodies only. It is sufficient to consider just a line on the rigid body (vector $\mathbf{AB}$, for example). Since the actual boundaries of the body does not influence the kinematics, the rigid body in plane motion is to be regarded as a large plane which embraces any desired point in the plane.
Vector Loops of a Mechanism

\[ A_0B_0 = a_1, \quad AA_0 = a_2, \quad AB = a_3, \quad B_0B = a_4 \]

\[ \theta_{12}, \theta_{13}, \theta_{14} \] are the position variables.

\[ A_0A + AB = A_0B \] (for open kinematic chain 1,2,3)

\[ A_0B_0 + B_0B = A_0B \] (for open kinematic chain 1,4)

\[ A_0A + AB = A_0B_0 + B_0B \] \textit{loop closure equation (vector loop equation)}

\[ a_2 \cos \theta_{12} \bar{i} + a_2 \sin \theta_{12} \bar{j} + a_3 \cos \theta_{13} \bar{i} + a_3 \sin \theta_{13} \bar{j} = a_1 \bar{i} + a_4 \cos \theta_{14} \bar{i} + a_4 \sin \theta_{14} \bar{j} \]

\[ a_2 \cos \theta_{12} + a_3 \cos \theta_{13} = a_1 + a_4 \cos \theta_{14} \]

\[ a_2 \sin \theta_{12} + a_3 \sin \theta_{13} = a_4 \sin \theta_{14} \]

\[ a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = a_1 + a_4 e^{i\theta_{14}} \] \textit{Loop Closure equation in complex numbers}
\[ \theta_{12}, \theta_{13}, s_{14} \] are the position variables.

\[ A_0A + AB = A_0B \] (for open kinematic chain 1,2,3)

\[ A_0B \] (for open kinematic chain 1,4)

\[ A_0A + AB = A_0B \] \textit{loop closure equation (vector loop equation)}

\[ a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = s_{14} + ic \] \textit{Loop Closure equation in complex numbers}
$\theta' = \theta_{13} - \pi$

$A_0 \overrightarrow{A} = A_0 \overrightarrow{B} + \overrightarrow{B} \overrightarrow{A}$

loop closure equation (vector loop equation)

$a_2 e^{i\theta_{12}} = s_{14} + ic + a_3 e^{i\theta'_{13}}$

Loop Closure equation in complex numbers
Inverted Slider Crank Mechanism

\[
\begin{align*}
\overrightarrow{AA_0} &= a_2, \quad \overrightarrow{B_0C} = a_4, \quad \overrightarrow{A_0B_0} = a_1 \\
\overrightarrow{A_0A} &= \overrightarrow{A_0B_0} + \overrightarrow{B_0C} + \overrightarrow{CA} \\
a_2 e^{i\theta_{12}} &= a_1 + a_4 e^{i\theta_{14}} + s_{43} e^{i(\theta_{14} + \alpha_4)}
\end{align*}
\]
Two loops!

\[ A_0D_0 = a_1, \quad A_0A = a_2, \quad AC = a_3, \quad AB = b_3, \quad CB = c_3, \quad EC = a_4, \quad D_0E = a_5, \quad D_0D = b_5, \quad ED = c_5, \quad BD = a_6 \]

\[ A_0A + AC = A_0D_0 + D_0E + EC \]
\[ a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = a_1 + a_5 e^{i\theta_{15}} + a_4 e^{i\theta_{14}} \]

\[ A_0A + AB = A_0D_0 + D_0D + DB \]
\[ a_2 e^{i\theta_{12}} + b_3 e^{i(\theta_{13} + \alpha_3)} = a_1 + b_5 e^{i(\theta_{15} + \alpha_5)} + a_6 e^{i\theta_{16}} \]
A third Loop equation

\[ EC + CB = ED + DB \]

\[ a_4 e^{i \theta_{14}} + c_3 e^{i (\theta_{13} + \beta_3)} = c_5 e^{i (\theta_{15} + \beta_5)} + a_6 e^{i \theta_{16}} \]

\[ A_0 A + AC = A_0 D_0 + D_0 E + EC \]

\[ EC + CB = ED + DB \]

Add \[ A_0 A + AC + EC + CB = D_0 E + EC + ED + DB \]

\[ AC + CB = AB \]

Similarly \[ D_0 E + ED = D_0 D \]

These are not loop closure Eqns.

There are only 2 independent loops

\[ A_0 A + AB = A_0 D_0 + D_0 D + DB \]
Door Mechanism for a Dishwasher
A₀A = a₂; A₀G = b₂; GF = a₃; BA = a₆; B₀B = a₅; B₀C = b₅; B₀E = c₅; EF = a₅; CD = a₅; <BB₀C = α₅; <BB₀E = β₅

Vector Loop Equations:

A₀A + AB = A₀B₀ + B₀B
A₀G + GF = A₀B₀ + B₀E + EF
B₀C + CD = B₀D

Loop Closure equations in complex numbers:

\[ a₂e^{iθ₂} + a₆e^{iθ₆} = -a₁ + ib₁ + a₃e^{iθ₃} \]
\[ b₂e^{i(θ₂ + α₂)} + a₃e^{iθ₃} = -a₁ + ib₁ + c₅e^{i(θ₅ + β₅)} + a₄e^{iθ₄} \]
\[ b₅e^{i(θ₅ + α₅)} + a₇e^{iθ₇} = c₁ + is₁₈ \]
$A_0A + AB = A_0B$

$A_0A + AC = A_0C$

These are not loop equations

$AC + CB = AB$
Hints

1. Only **one variable angle** must be used to define the angular orientation of a link.

2. Use $a_j$, $b_j$, $c_j$ for the fixed link lengths and $\alpha_j$, $\beta_j$, $\gamma_j$ for the fixed angles $\theta_{1j}$ for the variable link angles and $s_{jk}$ for the variable lengths.

3. Beware of special positions at which the mechanism is drawn.
Euler’s Equation of Polyhedra

\[ j = (l - 1) + L \]

or

\[ L = j - l + 1 \]  

(Euler’s Equation of polyhedra)

\[ j = \text{the number of joints in the open kinematic chain} + \text{the number of joints removed}. \]
Graphical Solution:

If you are given 4 links, you can combine them in 8 different ways.

Given one form of assembly determine the position of the links when the independent parameter changes its value from $\theta_{12}$ to $\theta_{12}'$.

$$A_0A + AB = A_0B_0 + B_0B$$

$$A_0A' + AB' = A_0B_0' + B_0B'$$
\[ A_0 B_0 + A_0 A' = B_0 A' \]

Both vectors are known

Solve the vector equation

\[ B_0 A' + A'B' = B_0 B' \]

The magnitudes of the three vectors are known

What about \( B'' \)???
Grashof’s Rule

1. The link may have a full rotation about the fixed axis (crank)
2. The link may oscillate (swing) between two limiting angles (rocker).

3 possibilities for a four-bar mechanism:

i) Both of the links connected to the fixed link can have a full rotation. This type of four-bar is called "double-crank" or "drag-link."

j) Both of the links connected to the fixed link can only oscillate. This type of four-bar is called "double-rocker."

k) One of the links connected to the fixed link oscillates while the other has a full rotation. This type of four-bar is called "crank-rocker".
l= length of the longest link
s= length of the shortest link
p,q = length of the two intermediate links

\[ l=830, \, s=216, \, p=485, \, q=581 \]
\[ 830 + 216 = 1046 \quad < \quad 485 + 581 = 1066 \]

IF \( l + s < p + q \)

a), b) Two different crank-rocker mechanisms are possible. In each case the shortest link is the crank, the fixed link is either of the adjacent links.

c) One double-crank (drag-link) is possible when the shortest link is the frame.

d) One double-rocker mechanism is possible when the link opposite the shortest link is the frame.
\[ l + s < p + q \]
\[ l + s > p + q \]

Only double-rocker mechanisms are possible (four different mechanisms, depending on the fixed link).

\[ l=829, \ s=216, \ p=485, \ q=415 \]

\[ 829 + 216 = 1045 > 485 + 415 = 900 \]

Grashof’s rule does not depend on how the links with different size are connected to each other.
\[ l + s = p + q \]

Same as \( l + s < p + q \) but “Change point” exists.
A position of the mechanism where all the joints are colinear (lie on a straight line)
Paralelogram Linkage

Deltoid Linkage

$AB = 3 \text{m}$

$A_B = A_A$
**Dead-Center Positions of Crank-Rocker Mechanisms**

ψ = swing angle
φ = corresponding crank rotation
β = initial crank angle

Determine ψ and φ using cosine theorem.

\[ TR = \frac{\text{time it takes for forward stroke}}{\text{time it takes for reverse stroke}} = \frac{\phi}{360^\circ - \phi} \]
Transmission angle is a kinematic quantity which gives us an idea on how well the force is transmitted.

\[
\cos \mu = \frac{a_4^2 + a_3^2 - a_1^2 - a_2^2}{2a_3a_4} + \frac{a_1a_2}{a_3a_4} \cos \theta_{12}
\]

\[
\cos \mu_{\text{min}}^{\text{max}} = \frac{a_4^2 + a_3^2 - a_1^2 - a_2^2}{2a_3a_4} \pm \frac{a_1a_2}{a_3a_4}
\]
Examples:

Slider-Crank Mechanism

Inverted Slider-Crank Mechanism

For the full rotatability of the crank:

Eccentricity < \((a_3-a_2)\) and \(a_3 > a_2\)
Function routines to determine the magnitude and the angle of a vector (rectangular to polar conversion)

**Function Mag(X, Y)**

' returns the magnitude of the vector

\[
\text{Mag} = \sqrt{X^2 + Y^2}
\]

End Function

**Function Ang(X, Y)**

' returns the angle the vector makes wr to +ve x axis

Dim AA, Pi

Pi = 4 * Atn(1)

If Abs(X) > eps Then
    AA = Atn(Y / X)
    If X < 0 Then
        AA = AA + Pi
    Else:
        If Y < 0 Then AA = AA + 2 * Pi
    End If
Else:
    If Y > 0 Then AA = Pi/2
    Else AA = -Pi / 2
End If

Ang = AA
End Function
Function routines to solve an unknown angle or length of a triangle using cosine theorem

Function AngCos(u1, u2, Opposite)
'returns the angle alfa
Dim U
U = (u1 * u1 + u2 * u2 - Opposite * Opposite) / (2 * u1 * u2)
AA = Acos(U)
AngCos = AA
End Function

Function MagCos(u1, u2, Angle)
' returns the length of the side opposite to the side
MagCos = Sqr(u1 * u1 + u2 * u2 - 2 * u1 * u2 * Cos(Angle))
End Function

Function Acos(X)
   Acos = Atn(-X / Sqr(-X * X + 1)) + 2 * Atn(1)
End Function

Function Asin(X)
   Asin = Atn(X / Sqr(-X * X + 1))
End Function
Stepwise Solution

Write a set of equations which can be solved in steps to yield a complete analysis of the mechanism

Or

Derive an algorithm to perform a complete position analysis

Example: Four-bar

\[ B_0A = s_x + is_y = s < \phi \]

\[ s_x = a_2 \cos(\theta_{12}) - a_1 \quad (1) \]

\[ s_y = a_2 \sin(\theta_{12}) \quad (2) \]

\[ s = \sqrt{s_x^2 + s_y^2} \quad (3) \]

\[ \phi = a \tan^{-1}(s_x, s_y) \quad (4) \]

\[ \psi = \cos^{-1} \left[ \left( \frac{a_4^2 + s^2 - a_2^2}{2a_4} \right) / 2a_4a_4 \right] \quad (5) \]

\[ \mu = \pm \cos^{-1} \left[ \frac{a_3^2 + a_4^2 - s^2}{2a_3a_4} \right] \quad (6) \]

\[ \theta_{14} = \phi \pm \psi \quad (7) \]

\[ \theta_{13} = \theta_{14} - \mu \quad (8) \]
Given:
The link lengths $a_1, a_2, a_3, a_4$,
the configuration (config)
The input crank angle $\theta_{12}$
Determine: The position of the links
Solve equations 1-8
Config= +1 or -1(cross configuration)

### Function FourBar(Crank, Coupler, Rocker, Fixed, Config, Theta)

```vbnet
Function FourBar(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim S, Fi, Si As Double
Dim sx, sy As Double
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
S = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
FourBar = Fi - Config * Si
End Function
```
Function FourBrCoupler(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim S, Fi, Si, Mu As Double
Dim sx, sy, Theta4 As Double
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
S = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Theta4 = Fi - Config * Si
Mu = AngCos(Coupler, Rocker, S)
FourBrCoupler = Theta4 - Mu
End Function
Function FourBar2(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim s, Fi, Si As Double
Dim sx, sy As Double
Dim A(2)
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
s = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Mu = AngCos(Coupler, Rocker, s)
A(1) = Fi - Config * Si
A(0) = A(1) - Mu
FourBar2 = A
End Function
Example: Slider-Crank Mechanism

\[ \phi = \sin^{-1}\left[ \frac{a_2 \sin \theta_{12} - a_1}{a_3} \right] \]

If Config = +1 then \( \theta_{13} = \pi - \phi \)
If Config = -1 then \( \theta_{13} = \phi \)
\( s_{14} = a_2 \cos \theta_{12} - a_3 \cos \theta_{13} \)
Example: Slider-Crank mechanism

Function SliderCrank(Crank, Coupler, Eccentricity, Config, Theta)
Dim Fi As Double
Fi = Asin((Crank * Sin(Theta) - Eccentricity) / Coupler)
If Config = 1 Then Fi = 4 * Atn(1) - Fi
SliderCrank = Crank * Cos(Theta) - Coupler * Cos(Fi)
End Function
Function with double argument

Function FourBar2(Crank, Coupler, Rocker, Fixed, Config, Theta)

Dim s, Fi, Si As Double
Dim sx, sy As Double
Dim A(2)
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
s = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Mu = AngCos(Coupler, Rocker, s)
A(1) = Fi - Config * Si
A(0) = A(1) - Mu
FourBar2 = A
End Function
Function SliderCrank2(Crank, Coupler, Eccentricity, Config, Theta)
Dim Fi As Double
Dim A(2) As Double
Fi = Asin((Crank * Sin(Theta) - Eccentricity) / Coupler)
If Config = 1 Then Fi = 4 * Atn(1) - Fi
A(0) = Fi
A(1) = Crank * Cos(Theta) - Coupler * Cos(Fi)
SliderCrank2 = A
End Function