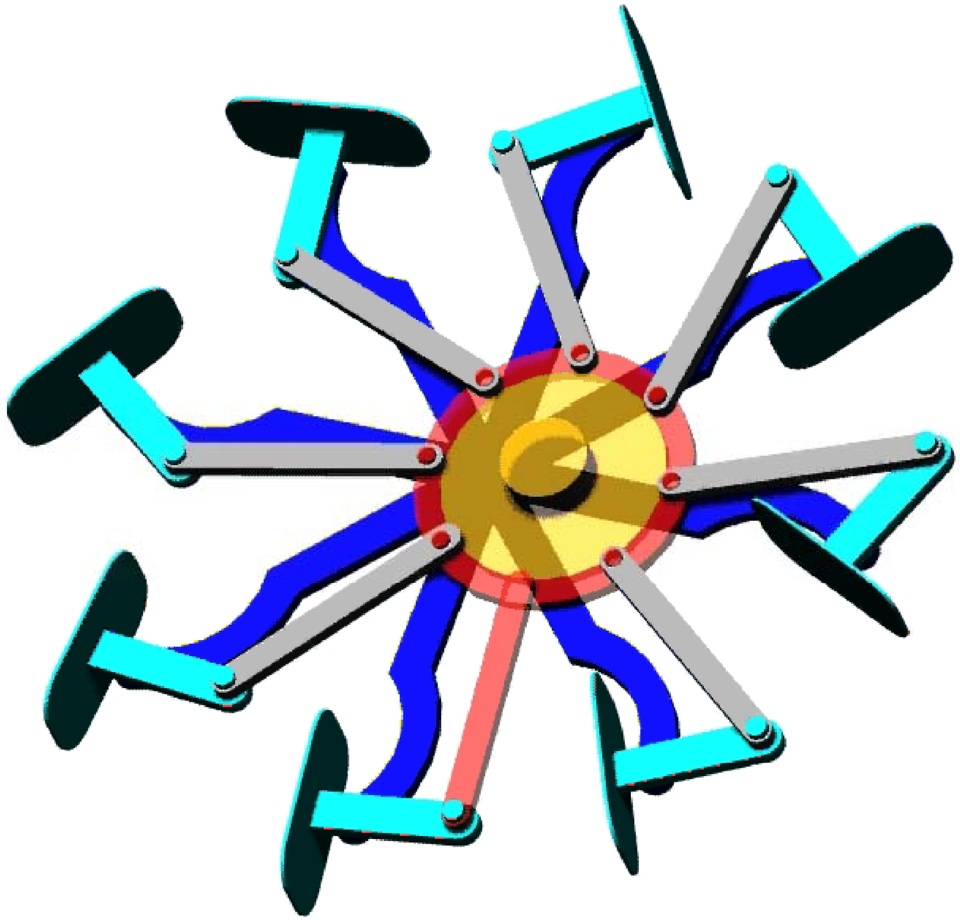


ME301 THEORY OF MACHINES 1- MECHANISMS

Some Important Remarks for the Exam:

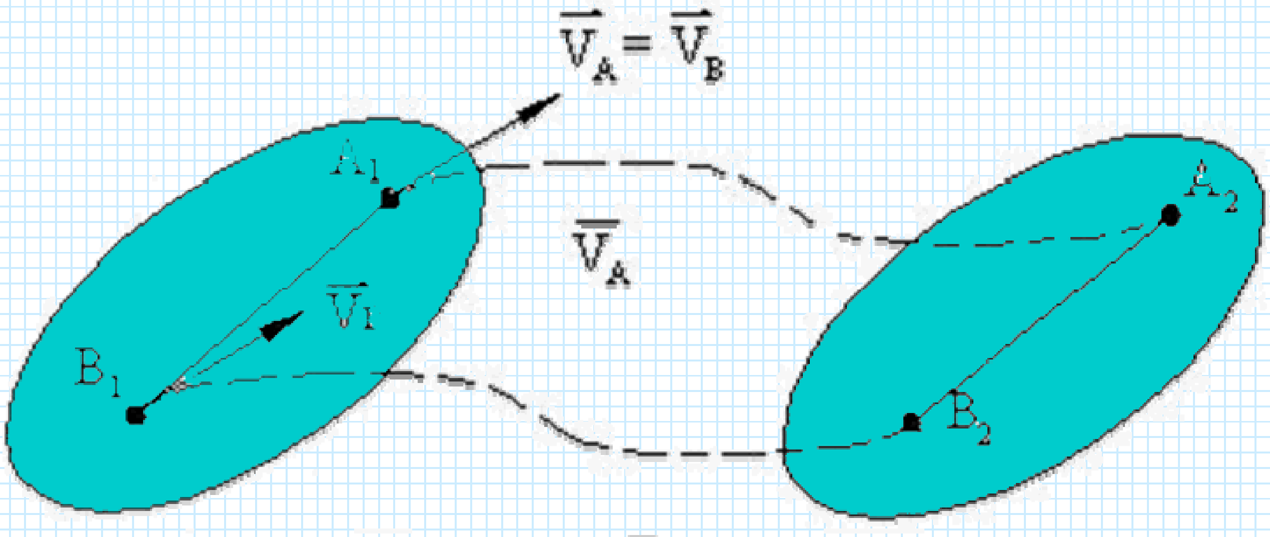
1. Use a_i, b_i,..., α_i, β_i, ... for the fixed link dimensions and angles respectively. i must refer to the link number on which that length or angle is to be measured. These angles or lengths have positive values only.
2. Use θ_{1j} for a variable angle that shows the angular position of link j with respect to a reference frame on link 1. Link 1 is always considered as the fixed link (or the refence link about which the motion of all the other links are to be measured). This angle must be a directed angle and must be considered posive when measured CCW.
3. You can use ϕ,ψ,ξ,... For intermediate variable angles that you may need during computation and s,u,t...for intermediate variable lengths.
4. Use A,B,C, ... For moving revolute joint axes. Do not use index. If you use index i, then i must refer to the link number where that point is located. Use subscript 0 for the fixed revolute joint axes: A_0, B_0, etc. If you have a point A then A_0 must be the center of the circle described by A.
5. When writing a complex number in polar form, simplify when you have an angle $\pi/2$ multiples.

$$e^{i(\theta+\pi/2)} = ie^{i\theta},\ e^{i(\theta-\pi/2)} = -ie^{i\theta},\quad e^{i(\theta+\pi)} = -e^{i\theta}\ ,\ e^{i(\theta+3\pi/2)} = -ie^{i\theta},\quad \text{etc}$$

Types of Plane Motion:

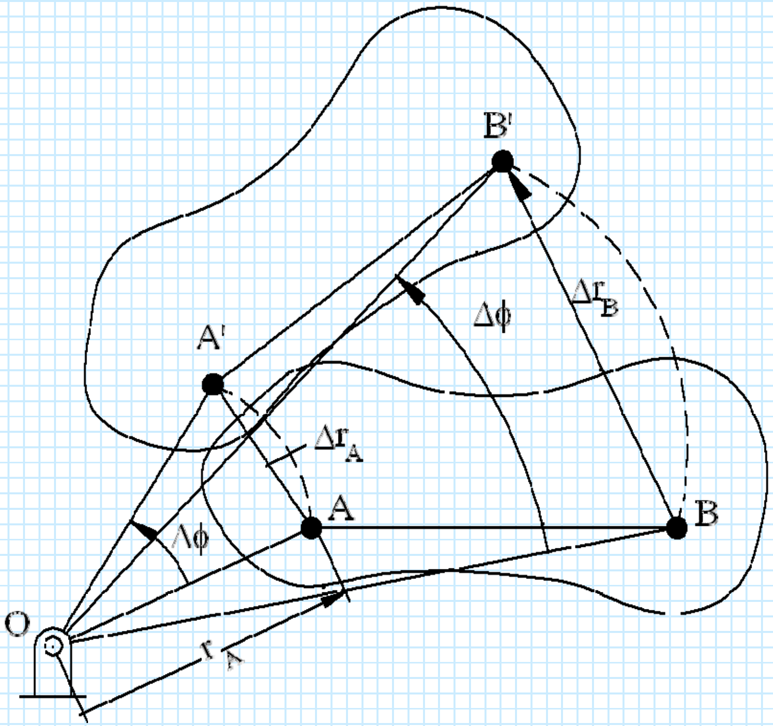
1. Translation of a rigid body
2. Rotation about a fixed axis
3. General Plane Motion

1. Translation



The velocity and acceleration of every point on the rigid body will be equal at each instant if the rigid body is in a translation.

2. Rotation about a fixed axis



$$\mathbf{V_A} = \omega \mathbf{x}\ \mathbf{r_A}$$

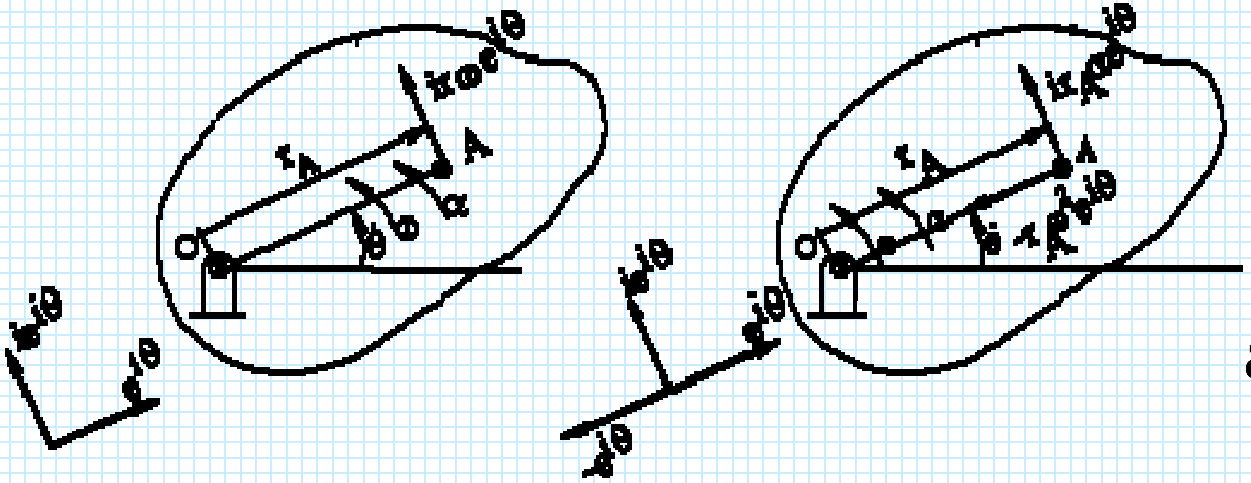
$$\mathbf{r_A} = r_A\, e^{i\theta}$$

$$\mathbf{V_A} = i\, r_A\, \omega\, e^{i\theta}$$

$$\frac{d\theta}{dt} = \omega$$

1. Points move on concentric circular arcs. (Center at O)
2. Velocity of a point is perpendicular to the line joining that point with the center of rotation.
3. Velocity magnitude is proportional to the distance from that point to the center of rotation (times the angular velocity of the rigid body)

2. Rotation about a fixed axis (Continued)

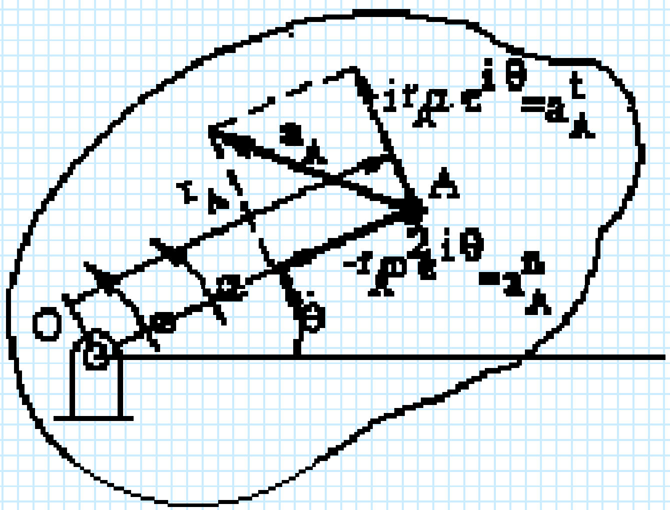


$$a_A = \frac{dv_A}{dt} = \frac{d}{dt}\left(ir_A e^{i\theta}\right)$$

$$= i\frac{d\omega}{dt}r_A e^{i\theta} - r\omega^2 e^{i\theta}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$a_A = i\alpha r_A e^{i\theta} - r\omega^2 e^{i\theta}$$



$$\mathbf{a_A} = \mathbf{a_A^t} + \mathbf{a_A^n}$$

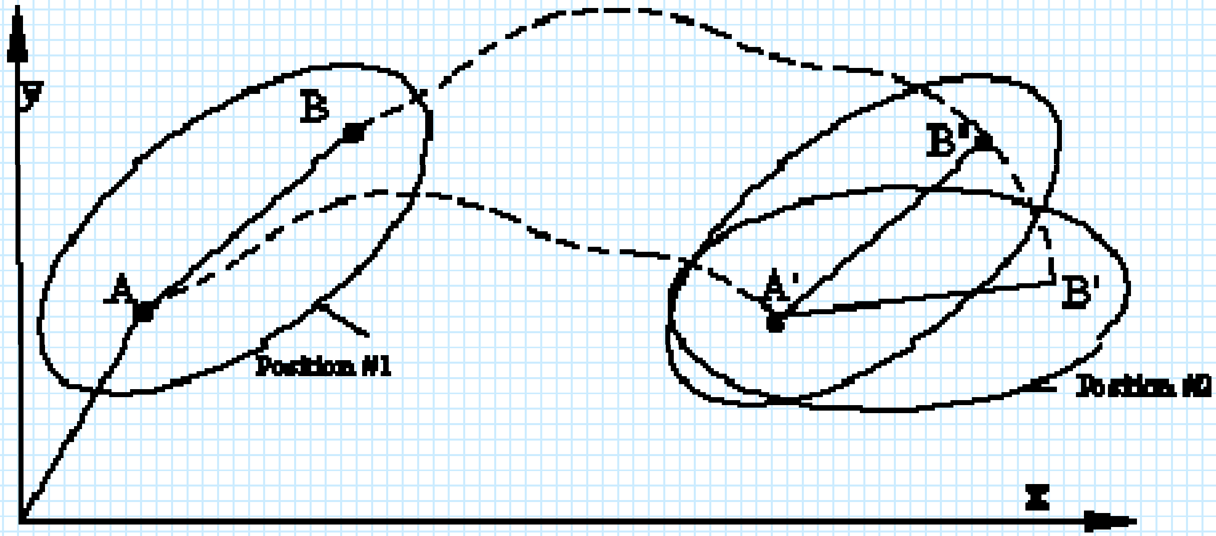
1. The velocity of every point is equal to the distance from that point to the axis of rotation times the angular velocity of the rigid body. The velocity vector is perpendicular to the line connecting the point to the axis of rotation and in the sense of angular velocity.

2. The normal acceleration of every point is equal to the distance from that point to the axis of rotation times the square of the angular velocity. The normal acceleration is always towards the axis of rotation along the line joining the point to the axis of rotation.

3. Tangential acceleration of every point is equal to the distance from that point to the axis of rotation times the angular acceleration of the rigid body. The tangential acceleration vector is perpendicular to the line connecting the point to the axis of rotation and in the sense of angular acceleration.

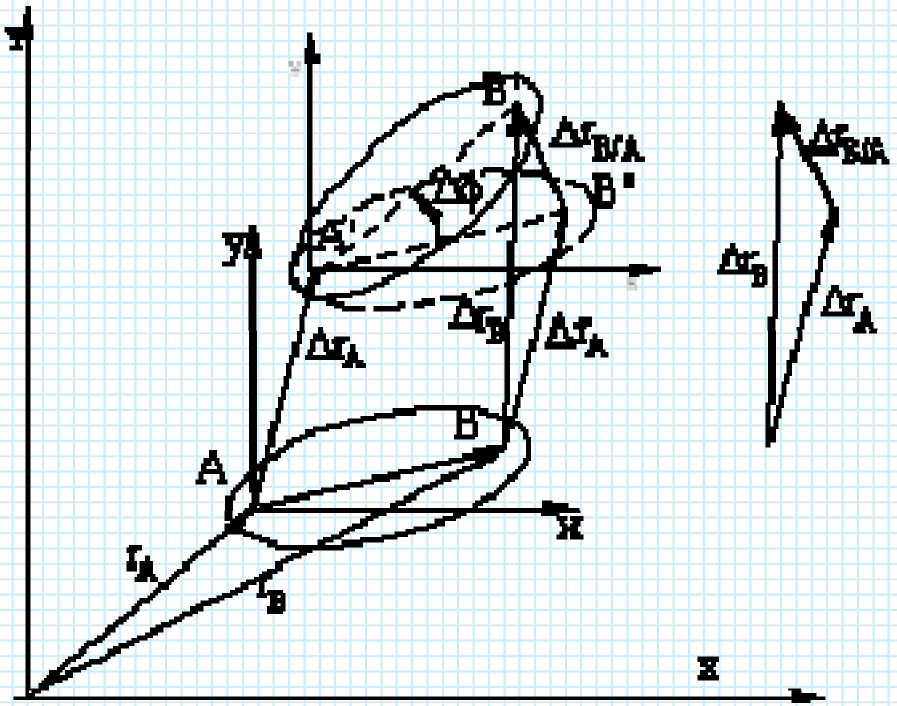
4. The acceleration of a point on the rigid body is the vectorial sum of the tangential and normal acceleration components.

3. General Plane Motion



Can be separated into two motions:

1. A translation from AB to A'B" (motion of A)
2. A ratation about point A' (**<u>relative motion</u>** wr to A)



$\Delta r_B = \Delta r_A + \Delta r_{B/A}$

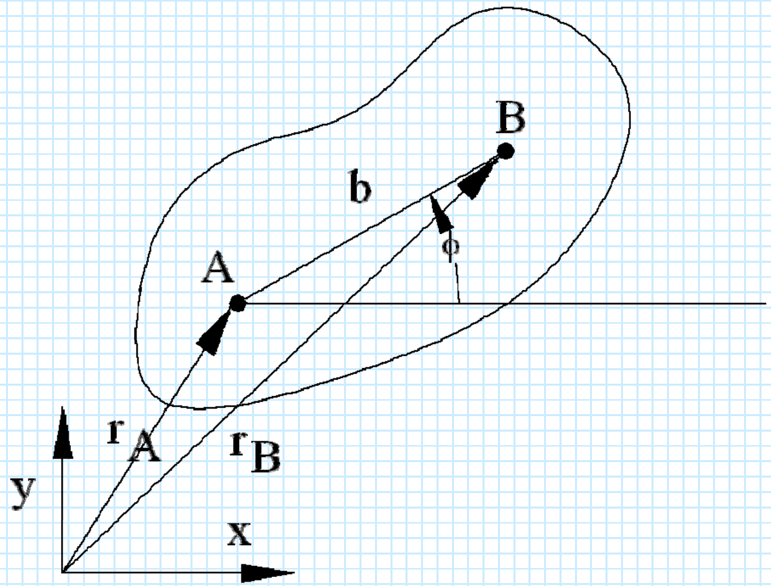
$V_B = V_A + V_{B/A}$

$V_{B/A} = \omega \times r_{B/A}$

$r_{B/A} = AB = r_B - r_A$

$V_{B/A}$ is the **relative velocity**

V_B and V_A are **absolute velocities**



In Complex numbers:

$$\mathbf{r_B} = \mathbf{r_A} + be^{i\phi}$$

Differentiate:

$$\mathbf{V_B} = \mathbf{V_A} + ib\omega e^{i\phi}$$

$$\mathbf{V_{B/A}} = ib\omega e^{i\phi}$$ **Relative velocity**

$$\mathbf{V_B} = \mathbf{V_A} + \mathbf{V_{B/A}}$$

Second Derivative:

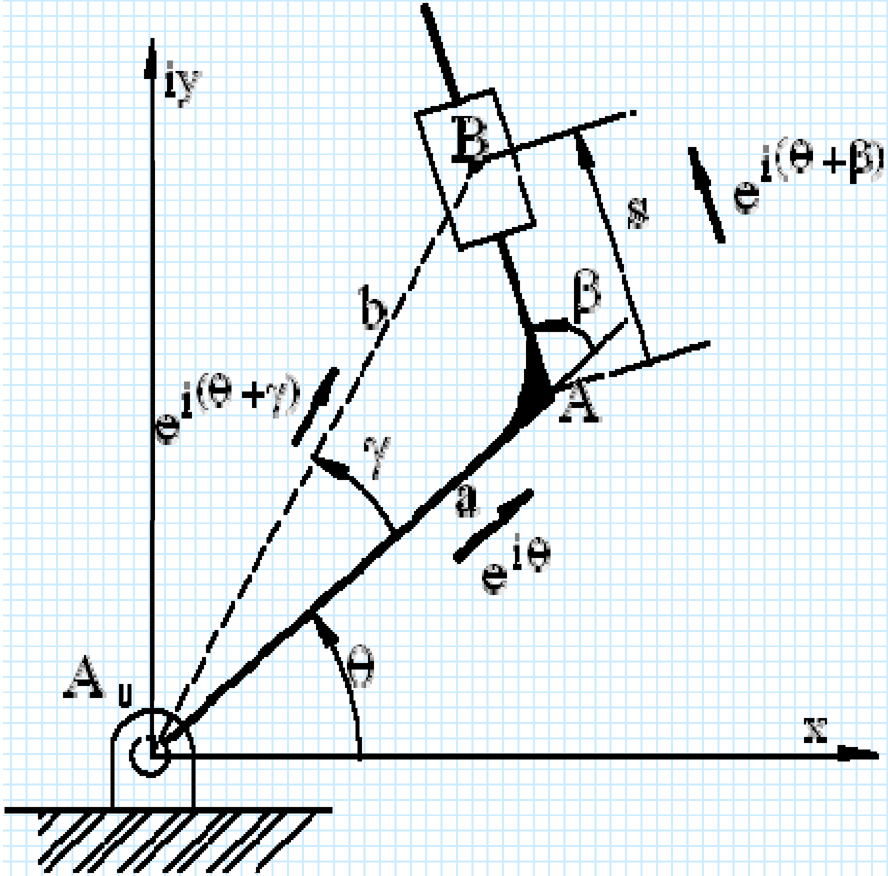
$$\mathbf{a_B} = \mathbf{a_A} + ib\alpha e^{i\phi} - b\omega^2 e^{i\phi}$$

$$\mathbf{a}^t_{B/A} = ib\alpha e^{i\phi}$$ **Tangential relative acceleration component**

$$\mathbf{a_B} = \mathbf{a_A} + \mathbf{a}^t_{B/A} + \mathbf{a}^n_{B/A}$$

$$\mathbf{a}^n_{B/A} = -b\omega^2 e^{i\phi}$$ ***normal relative acceleration component***

$$\mathbf{a_{B/A}} = \mathbf{a}^t_{B/A} + \mathbf{a}^n_{B/A}$$



$$r_{B_3} = ae^{i\theta} + se^{i(\theta+\beta)}$$

$$v_{B_3} = ia\omega e^{i\theta} + is\omega e^{i(\theta+\beta)} + \dot{s}\, e^{i(\theta+\beta)}$$

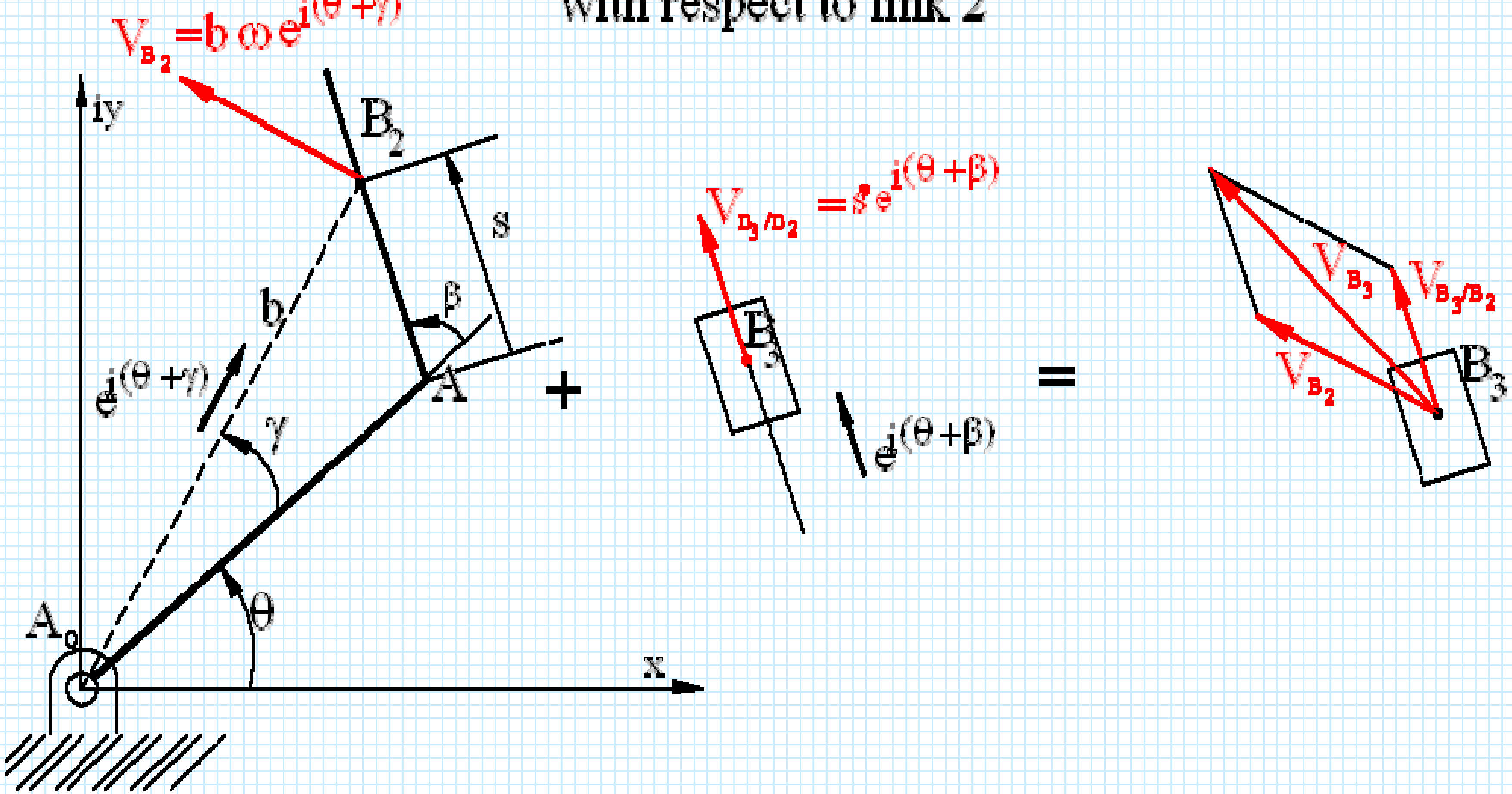
$$v_{B_3} = i\omega(a + se^{i\beta})e^{i\theta} + \dot{s}\, e^{i(\theta+\beta)}$$

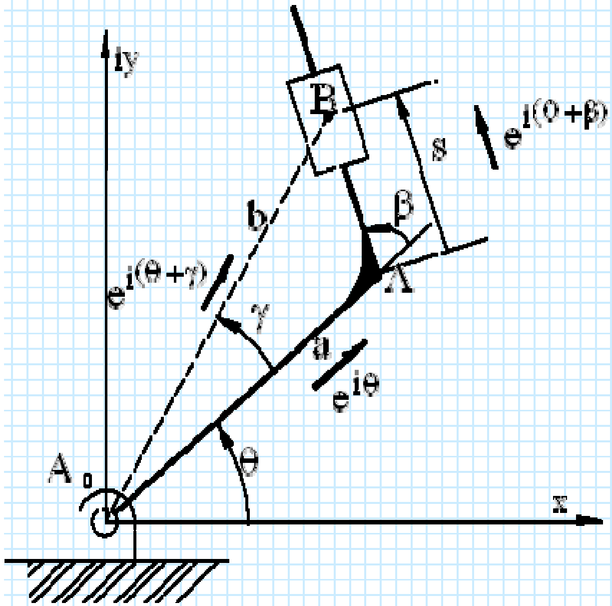
$$a + se^{i\beta} = be^{i\gamma}$$

$$V_{B_3} = i\omega be^{i(\theta+\gamma)} + \dot{s}\, e^{i(\theta+\beta)}$$

$V_{B3} = V_{B2} + V_{B3/2}$

Velocity of B_2 | Relative velocity of B_3 with respect to link 2 | Velocity of B_3





$$v_{B_3} = ia\omega e^{i\theta} + is\omega e^{i(\theta+\beta)} + \dot{s} e^{i(\theta+\beta)}$$

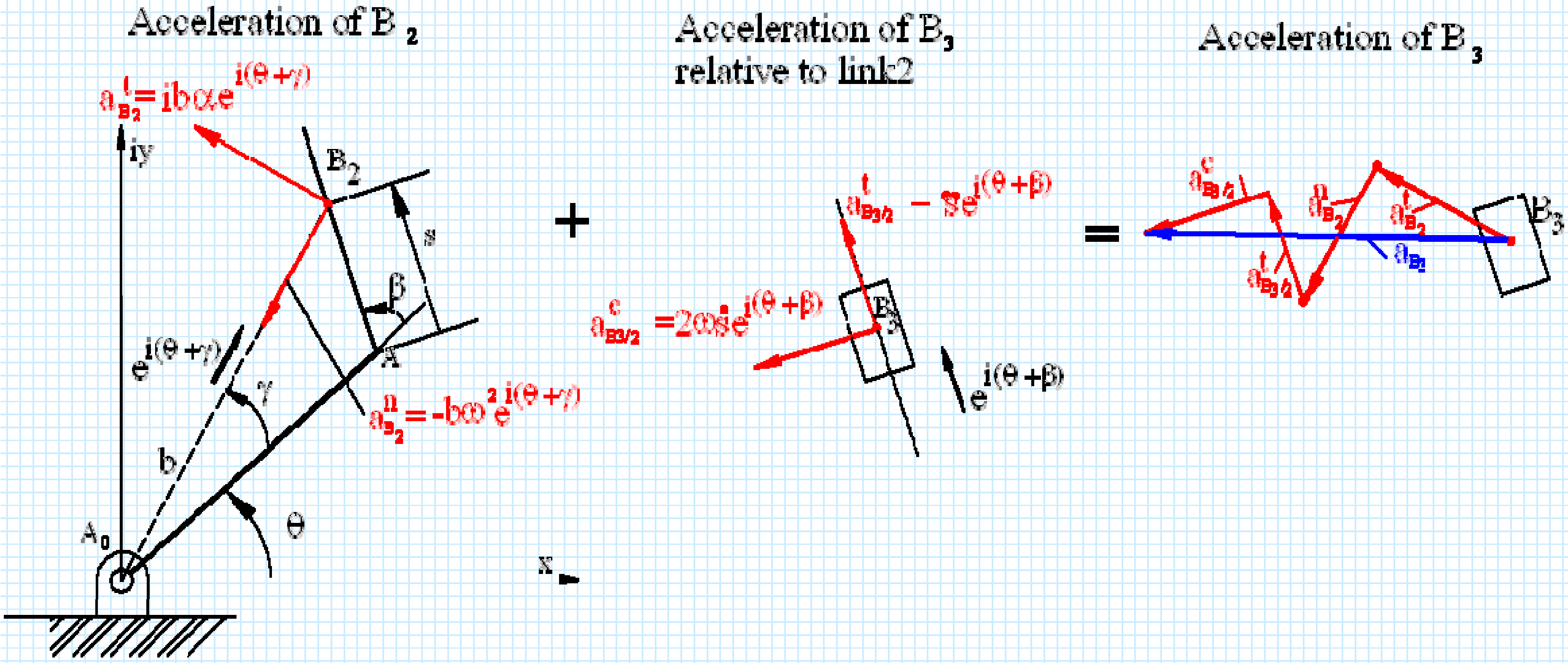
$$a_{B_3} = ia\alpha e^{i\theta} - a\omega^2 e^{i\theta} + is\alpha e^{i(\theta+\beta)} + i\dot{s}\omega e^{i(\theta+\beta)} - s\omega^2 e^{i(\theta+\beta)} + \ddot{s} e^{i(\theta+\beta)} + i\dot{s}\omega e^{i(\theta+\beta)}$$

$$a_{B_3} = i\alpha(a + se^{i\beta})e^{i\theta} - \omega^2(a + se^{i\beta})e^{i\theta} + 2i\dot{s}\omega e^{i(\theta+\beta)} + \ddot{s} e^{i(\theta+\beta)}$$

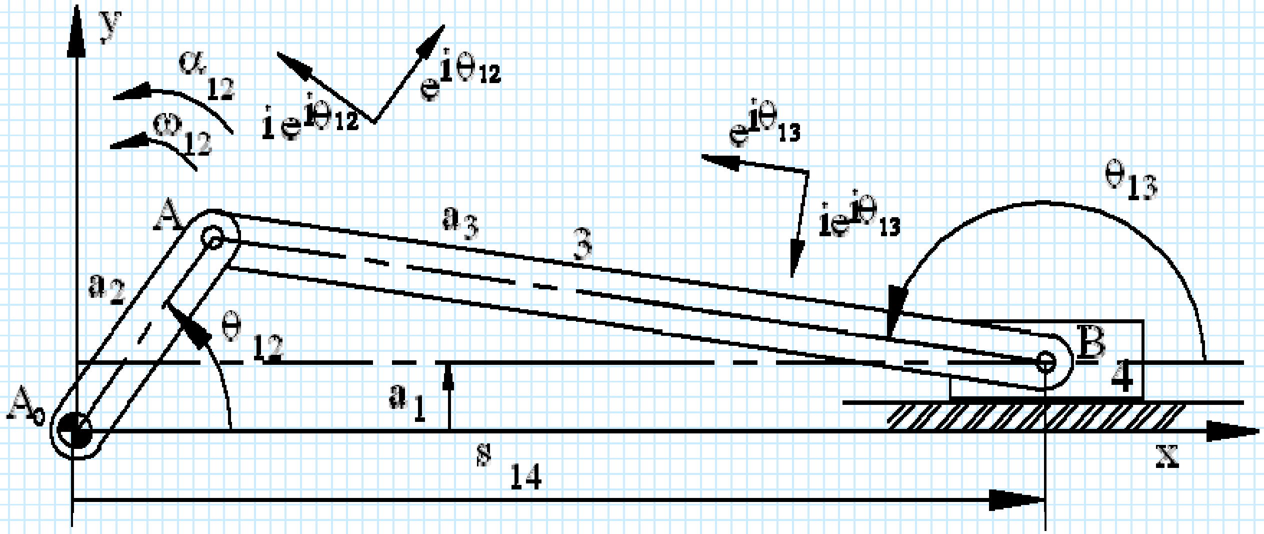
$$a + se^{i\beta} = be^{i\gamma}$$

$$a_{B_3} = i\alpha b e^{i(\theta+\gamma)} - \omega^2 b e^{i(\theta+\gamma)} + 2i\dot{s}\omega e^{i(\theta+\beta)} + \ddot{s} e^{i(\theta+\beta)}$$

$$a_{B_3} = a^t_{B_2} + a^n_{B_2} + a^c_{B_3/2} + a^t_{B_3/2}$$


Acceleration of B 2
Acceleration of B3 relative to link2
Acceleration of B 3
+
=
iy
x
B2
A0
A
b
s
β
γ
θ
B3

Example: Slider-Crank Mechanism



The loop closure and its complex conjugate :

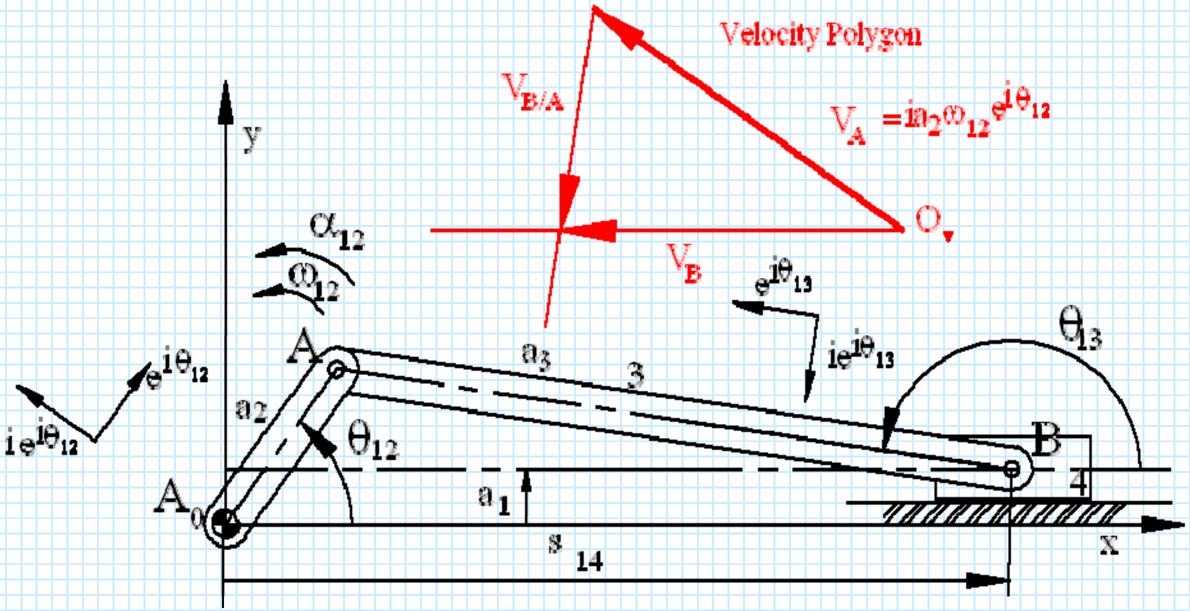
$$s_{14} + ia_1 + a_3 e^{i\theta_{13}} = a_2 e^{i\theta_{12}}$$

$$s_{14} - ia_1 + a_3 e^{-i\theta_{13}} = a_2 e^{-i\theta_{12}}$$

Velocity loop equation and its complex conjugate (obtained by taking the derivative of the loop equation wr to time):

$$\dot{s}_{14} + ia_3\omega_{13} e^{i\theta_{13}} = ia_2\omega_{12} e^{i\theta_{12}}$$

$$\dot{s}_{14} - ia_3\omega_{13} e^{-i\theta_{13}} = -ia_2\omega_{12} e^{-i\theta_{12}}$$



$$\dot{s}_{14} + ia_3\omega_{13}e^{i\theta_{13}} = ia_2\omega_{12}e^{i\theta_{12}}$$

$$\mathbf{V_B + V_{A/B} = V_A}$$

Or:

$$\dot{s}_{14} = ia_2\omega_{12}e^{i\theta_{12}} - ia_3\omega_{13}e^{i\theta_{13}}$$

$$\mathbf{V_B = V_A + V_{B/A}}$$

$$V_{B/A} = \mathbf{-\,V_{A/B}} = -ia_3e^{i\theta_{13}}$$

Analytical Solution:

If the position analysis is performed for a given θ_{12} , θ_{13} is known and ω_{12} is given

$$\dot{s}_{14} + ia_3\omega_{13}e^{i\theta_{13}} = ia_2\omega_{12}e^{i\theta_{12}}$$

$$\dot{s}_{14} - ia_3\omega_{13}e^{-i\theta_{13}} = -ia_2\omega_{12}e^{-i\theta_{12}}$$

Two complex equations in two unknowns (ω_{13} and $\dot{s}_{14}$)

$$\dot{s}_{14} = \frac{\begin{bmatrix} ia_2\omega_{12}e^{i\theta_{12}} & ia_3e^{i\theta_{13}} \\ -ia_2\omega_{12}e^{-i\theta_{12}} & -ia_3e^{-i\theta_{13}} \end{bmatrix}}{\begin{bmatrix} 1 & ia_3e^{i\theta_{13}} \\ 1 & -ia_3e^{-i\theta_{13}} \end{bmatrix}} = \frac{a_2a_3(e^{i(\theta_{12}-\theta_{13})} - e^{-i(\theta_{12}-\theta_{13})})}{-ia_3(e^{i\theta_{13}} + e^{-i\theta_{13}})}\omega_{12}$$

$$\dot{s}_{14} = -a_2\frac{\sin(\theta_{12} - \theta_{13})}{\cos(\theta_{13})}\omega_{12}$$

$$\omega_{13} = \frac{\begin{bmatrix} 1 & ia_2\omega_{12}e^{i\theta_{12}} \\ 1 & -ia_2\omega_{12}e^{-i\theta_{12}} \end{bmatrix}}{\begin{bmatrix} 1 & ia_3e^{i\theta_{13}} \\ 1 & -ia_3e^{-i\theta_{13}} \end{bmatrix}} = \frac{-ia_2(e^{i\theta_{12}} + e^{-i\theta_{12}})}{-ia_3(e^{i\theta_{13}} + e^{-i\theta_{13}})}\omega_{12} = \frac{a_2}{a_3}\frac{\cos\theta_{12}}{\cos\theta_{13}}\omega_{12}$$

Velocity Loop Equation $\dot{s}_{14} + ia_3\omega_{13}e^{i\theta_{13}} = ia_2\omega_{12}e^{i\theta_{12}}$

Acceleration loop equation is obtained by taking the derivative of the velocity loop equation wr. to time.

$$\ddot{s}_{14} + ia_3\alpha_{13}e^{i\theta_{13}} - a_3\omega_{13}^2 e^{i\theta_{13}} = ia_2\alpha_{12}e^{i\theta_{12}} - a_3\omega_{12}^2 e^{i\theta_{13}}$$

And its complex conjugate

$$\ddot{s}_{14} - ia_3\alpha_{13}e^{-i\theta_{13}} - a_3\omega_{13}^2 e^{-i\theta_{13}} = -ia_2\alpha_{12}e^{-i\theta_{12}} - a_3\omega_{12}^2 e^{-i\theta_{13}}$$

$$\vec{\mathbf{a}}_{\mathbf{B}} + \vec{\mathbf{a}}^{\mathbf{t}}_{\mathbf{A/B}} + \vec{\mathbf{a}}^{\mathbf{n}}_{\mathbf{A/B}} = \vec{\mathbf{a}}^{\mathbf{t}}_{\mathbf{A}} + \vec{\mathbf{a}}^{\mathbf{n}}_{\mathbf{A}}$$

The acceleration loop equations are linear in terms of the acceleration variables (α_{12}, α_{13} and $\ddot{s}_{14}$)

$$\alpha_{13} = \frac{\left(a_2\alpha_{12}\cos\theta_{12} - a_2\omega_{12}^2\sin\theta_{12} + a_3\omega_{13}^2\sin\theta_{13}\right)}{a_3\cos\theta_{13}}$$

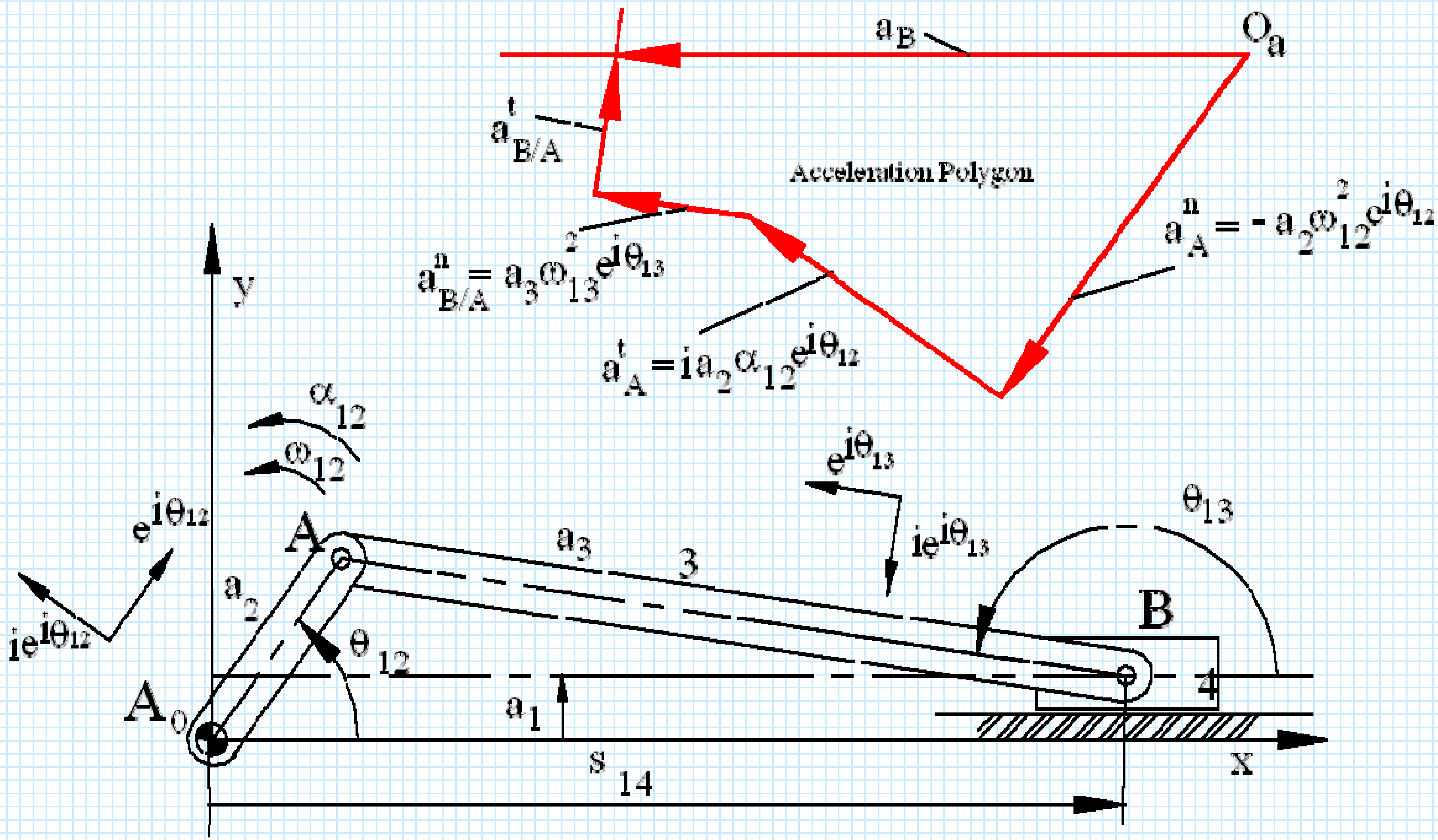
$$\ddot{s}_{14} = -a_2\alpha_{12}\sin\theta_{12} - a_2\omega_{12}^2\cos\theta_{12} + a_3\alpha_{13}\sin\theta_{13} + a_3\omega_{13}^2\cos\theta_{13}$$

Or:

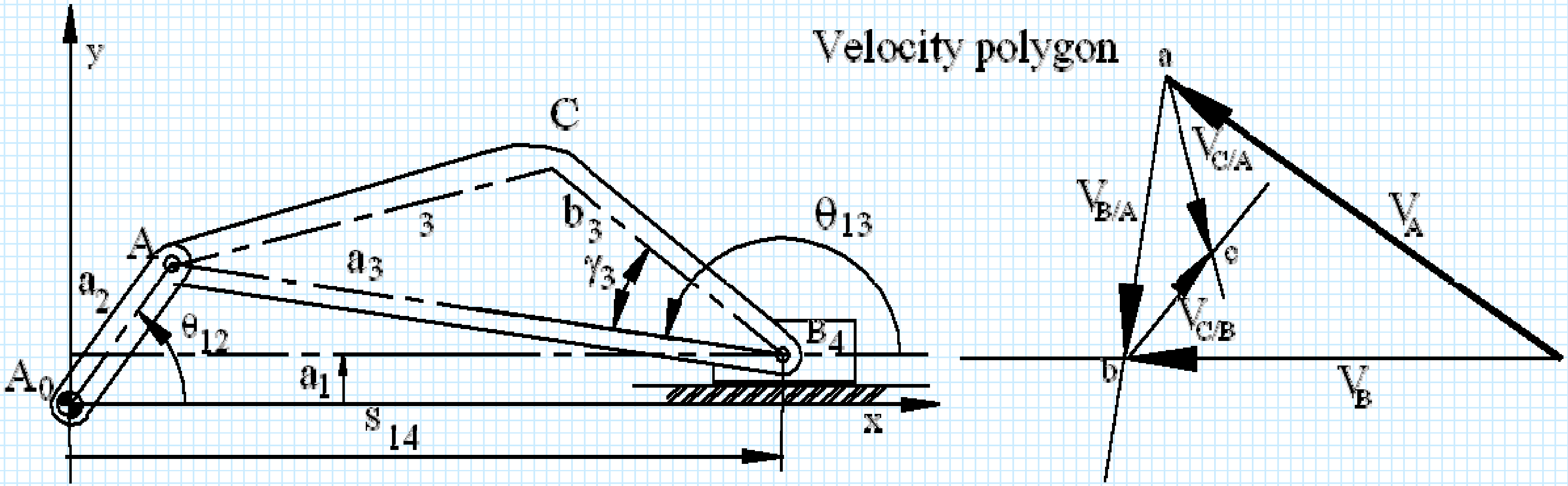
$$\omega_{13} = \frac{a_2}{a_3}\frac{\cos\theta_{12}}{\cos\theta_{13}}\omega_{12}$$

Differentiate:

$$\alpha_{13} = \frac{a_2}{a_3}\frac{1}{\cos^2\theta_{13}}\left[\cos\theta_{12}\cos\theta_{13}\alpha_{12} - \cos\theta_{13}\sin\theta_{12}\omega_{12}^2 + \cos\theta_{12}\sin\theta_{13}\omega_{12}\omega_{13}\right]$$



$$\vec{a}_B + \vec{a}^t_{A/B} + \vec{a}^n_{A/B} = \vec{a}^t_A + \vec{a}^n_A$$



Velocity and acceleration of a coupler point C can only be solved after the solution of loop equations (disp,velocity and acc. Loop equations)

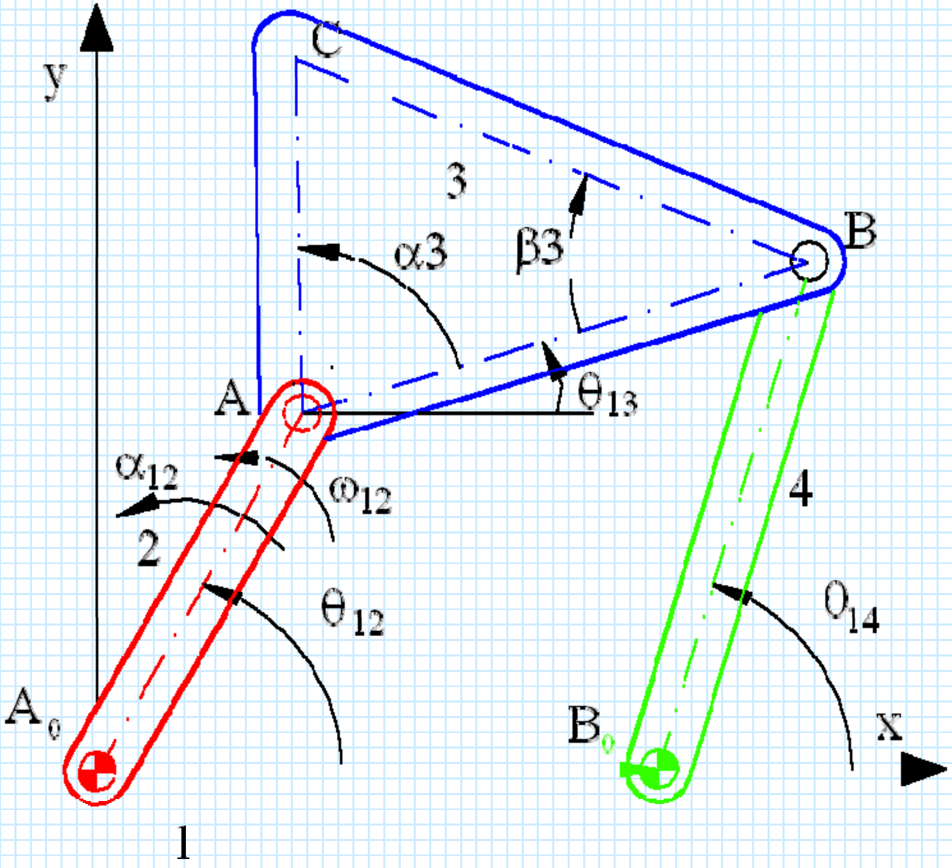
$$r_c = s_{14} + ia_1 + b_3 e^{i(\theta_{13}-\gamma_3)}$$

$$\vec{V}_c = \dot{s}_{14} + ib_3\omega_{13}e^{i(\theta_{13}-\gamma_3)}$$

$V_C = V_B + V_{C/B}$

$$\vec{a}_C = \ddot{s}_{14} + ib_3\alpha_{13}e^{i(\theta_{13}-\gamma_3)} - b_3\omega_{13}^2 e^{i(\theta_{13}-\gamma_3)}$$

$a_C = a_B + a^t_{C/B} + a^n_{C/B}$



The loop closure equation and its complex conjugate

$$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = a_1 + a_4e^{i\theta_{14}}$$

$$a_2e^{-i\theta_{12}} + a_3e^{-i\theta_{13}} = a_1 + a_4e^{-i\theta_{14}}$$

The first derivative of the loop closure equation (velocity loop equation)

$$ia_2\omega_{12}e^{i\theta_{12}} + ia_3\omega_{13}e^{i\theta_{13}} = ia_4\omega_{14}e^{i\theta_{14}}$$

$$-ia_2\omega_{12}e^{-i\theta_{12}} - ia_3\omega_{13}e^{-i\theta_{13}} = -ia_4\omega_{14}e^{-i\theta_{14}}$$

$$\mathbf{V_A + V_{B/A} = V_B}$$

or

$$ia_3\omega_{13}e^{i\theta_{13}} - ia_4\omega_{14}e^{i\theta_{14}} = -ia_2\omega_{12}e^{i\theta_{12}}$$

$$-ia_3\omega_{13}e^{-i\theta_{13}} + ia_4\omega_{14}e^{-i\theta_{14}} = +ia_2\omega_{12}e^{-i\theta_{12}}$$

Two linear equations in two unknowns ω_{13} **and** ω_{14}.

$$\omega_{13} = \frac{\begin{bmatrix} -a_2\omega_{12}e^{i\theta_{12}} & -a_4e^{i\theta_{14}} \\ -a_2\omega_{12}e^{-i\theta_{12}} & -a_3e^{-i\theta_{14}} \end{bmatrix}}{\begin{bmatrix} a_3e^{i\theta_{13}} & -a_4e^{i\theta_{14}} \\ a_3e^{-i\theta_{13}} & -a_4e^{-i\theta_{14}} \end{bmatrix}} = \frac{a_2a_4(e^{i(\theta_{12}-\theta_{14})} - e^{-i(\theta_{12}-\theta_{14})})}{a_3a_4(e^{i(\theta_{14}-\theta_{13})} + e^{-i(\theta_{14}-\theta_{13})})}\omega_{12}$$

$$\omega_{13} = \frac{a_2}{a_3}\frac{\sin(\theta_{12}-\theta_{14})}{\sin(\theta_{14}-\theta_{13})}\omega_{12}$$

$$\omega_{14} = \frac{\begin{bmatrix} a_3e^{i\theta_{13}} & -a_2\omega_{12}e^{i\theta_{12}} \\ a_3e^{-i\theta_{13}} & -a_2\omega_{12}e^{-i\theta_{12}} \end{bmatrix}}{\begin{bmatrix} a_3e^{i\theta_{13}} & -a_4e^{i\theta_{14}} \\ a_3e^{-i\theta_{13}} & -a_4e^{-i\theta_{14}} \end{bmatrix}} = \frac{a_3a_2(e^{i(\theta_{12}-\theta_{13})} - e^{-i(\theta_{12}-\theta_{13})})}{a_3a_4(e^{i(\theta_{14}-\theta_{13})} + e^{-i(\theta_{14}-\theta_{13})})}\omega_{12}$$

$$\omega_{14} = \frac{a_2}{a_4}\frac{\sin(\theta_{12}-\theta_{13})}{\sin(\theta_{14}-\theta_{13})}\omega_{12}$$

Displacement and velocity of coupler point C

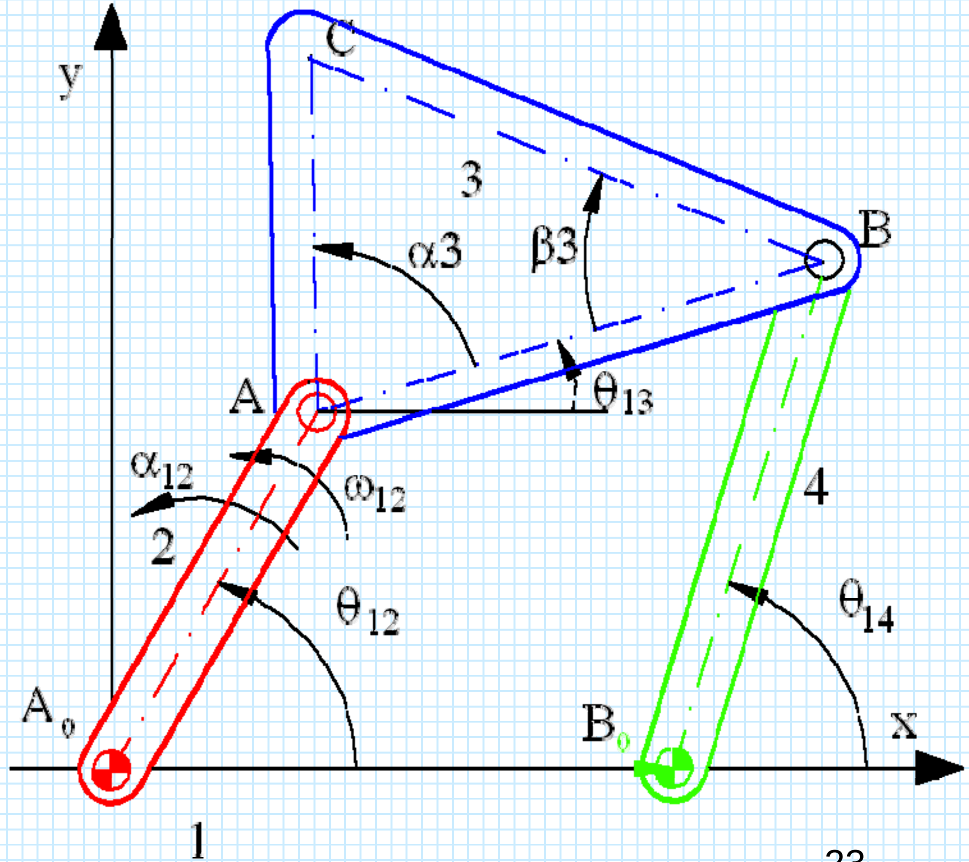
$$\vec{r}_c = a_2 e^{i\theta_{12}} + b_3 e^{i(\theta_{13}+a_3)}$$

$$\vec{V}_C = \dot{x}_C + \dot{y}_C = i a_2 \omega_{12} e^{i\theta_{12}} + i b_3 \omega_{13} e^{i(\theta_{13}+a_3)}$$

$$\mathbf{V_C} = \mathrm{V_c} e^{i\gamma}$$

$$\mathrm{V_C} = \sqrt{\left(\dot{x}_C^{\,2} + \dot{y}_c^{\,2}\right)}$$

$$\gamma = \tan^{-1}(\dot{x}_c, \dot{y}_c)$$



Acceleration Analysis:

Derivative of the velocity loop equation (or the second derivative of the loop equation w.r to time gives us acceleration loop equation.

$$ia_2\alpha_{12}e^{i\theta_{12}} - a_2\omega_{12}^2 e^{i\theta_{12}} + ia_3\alpha_{13}e^{i\theta_{13}} - a_3\omega_{13}^2 e^{i\theta_{13}} = ia_4\alpha_{14}e^{i\theta_{14}} - a_4\omega_{14}^2 e^{i\theta_{14}}$$

$$-ia_2\alpha_{12}e^{-i\theta_{12}} - a_2\omega_{12}^2 e^{-i\theta_{12}} - ia_3\alpha_{13}e^{-i\theta_{13}} - a_3\omega_{13}^2 e^{-i\theta_{13}} = -ia_4\alpha_{14}e^{-i\theta_{14}} - a_4\omega_{14}^2 e^{-i\theta_{14}}$$

$$a_A^t + a_A^n + a_{B/A}^t + a_{B/A}^n = a_B^t + a_B^n$$

$$ia_3\alpha_{13}e^{i\theta_{13}} + ia_4\alpha_{14}e^{i\theta_{14}} = -ia_2\alpha_{12}e^{i\theta_{12}} + a_2\omega_{12}^2 e^{i\theta_{12}} + a_3\omega_{13}^2 e^{i\theta_{13}} - a_4\omega_{14}^2 e^{i\theta_{14}}$$

$$-ia_3\alpha_{13}e^{-i\theta_{13}} - ia_4\alpha_{14}e^{-i\theta_{14}} = ia_2\alpha_{12}e^{-i\theta_{12}} + a_2\omega_{12}^2 e^{-i\theta_{12}} + a_3\omega_{13}^2 e^{-i\theta_{13}} - a_4\omega_{14}^2 e^{-i\theta_{14}}$$

Two linear equations in terms of two acceleration variables α_{13}, α_{13} (as unknown).

$$\alpha_{13} = \frac{1}{\sin(\theta_{14}-\theta_{13})}\left[\frac{a_2}{a_3}\omega_{12}^2\cos(\theta_{12}-\theta_{14}) - \frac{a_4}{a_3}\omega_{14}^2 + \frac{a_2}{a_3}\alpha_{12}\sin(\theta_{12}-\theta_{14}) + \omega_{13}^2\cos(\theta_{13}-\theta_{14})\right]$$

and

$$\alpha_{14} = \frac{1}{\sin(\theta_{14}-\theta_{13})}\left[\frac{a_2}{a_4}\omega_{12}^2\cos(\theta_{12}-\theta_{13}) + \frac{a_2}{a_4}\alpha_{12}\sin(\theta_{12}-\theta_{13}) - \omega_{14}^2\cos(\theta_{13}-\theta_{14}) + \frac{a_3}{a_4}\omega_{13}^2\right]$$

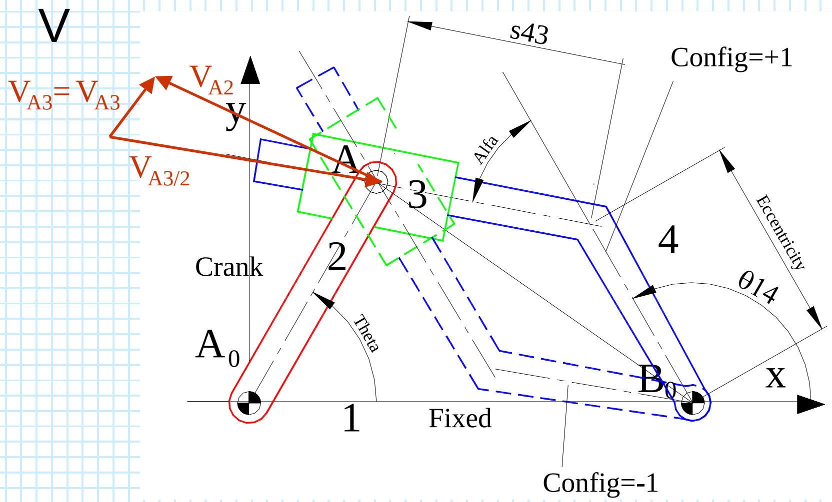
Acceleration of point C (second derivative of the position vector or the derivative of the velocity vector)

$$\vec{a}_C = -a_2\omega_{12}^2 e^{i\theta_{12}} + i a_2\alpha_{12} e^{i\theta_{12}} - b_3\omega_{13}^2 e^{i(\theta_{13}+\alpha_3)} + i b_3\alpha_{13} e^{i(\theta_{13}+\alpha_3)}$$

$$a_{Cx} = -a_2\omega_{12}^2\cos\theta_{12} - a_2\alpha_{12}\sin\theta_{12} - b_3\omega_{13}^2\cos(\theta_{13}+\alpha_3) - b_3\alpha_{13}\sin(\theta_{13}+\alpha_3)$$

$$a_{Cy} = -a_2\omega_{12}^2\sin\theta_{12} + a_2\alpha_{12}\cos\theta_{12} - b_3\omega_{13}^2\sin(\theta_{13}+\alpha_3) + b_3\alpha_{13}\cos(\theta_{13}+\alpha_3)$$

$Alfa = \beta_4$



$$a_2 e^{i\theta_{12}} = a_1 + a_4 e^{i\theta_{14}} + s_{43} e^{i(\theta_{14}+\beta_4)}$$

$$i a_2 \omega_{12} e^{i\theta_{12}} = i a_4 \omega_{14} e^{i\theta_{14}} + i s_{43} \omega_{14} e^{i(\theta_{14}+\beta_4)} + \dot{s}_{43} e^{i(\theta_{14}+\beta_4)}$$

Velocity Variables: $\omega_{14}, \dot{s}_{43}, \omega_{12}$

$$i\omega_{14}(a_4 + s_{43} e^{i\beta_4}) e^{i\theta_{14}} = i a_2 \omega_{12} e^{i\theta_{12}} - \dot{s}_{43} e^{i(\theta_{14}+\beta_4)}$$

$$a_4 + s_{43} e^{i\beta_4} = b_4 e^{i\gamma}$$

$$i b_4 \omega_{14} e^{i(\theta_{14}+\gamma)} = i a_2 \omega_{12} e^{i\theta_{12}} - \dot{s}_{43} e^{i(\theta_{14}+\beta_4)}$$

$$\vec{V}_{A_4} = \vec{V}_{A_3} = \vec{V}_{A_2} + \vec{V}_{A_{3/2}}$$

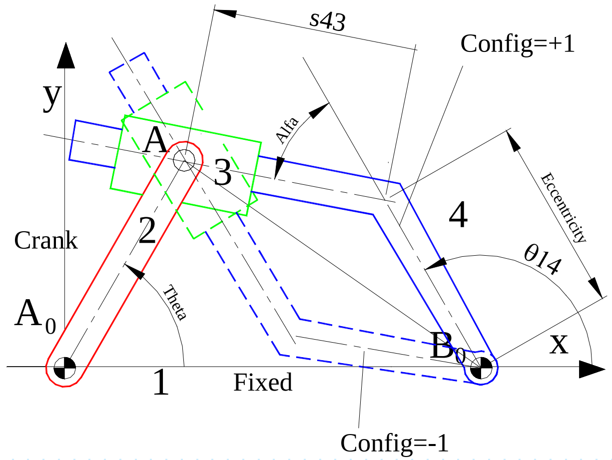
For graphical solution, have one unknown on each side of the equation.

For analytical solution, unknown velocity variables must be on one side of the equation

$$i\omega_{14}(a_4 + s_{43}e^{i\beta_4})e^{i\theta_{14}} + \dot{s}_{43}e^{i(\theta_{14}+\beta_4)} = ia_2\omega_{12}e^{i\theta_{12}}$$

V

$$-i\omega_{14}(a_4 + s_{43}e^{-i\beta_4})e^{-i\theta_{14}} + \dot{s}_{43}e^{-i(\theta_{14}+\beta_4)} = -ia_2\omega_{12}e^{-i\theta_{12}}$$



$$\begin{bmatrix} i(a_4 + s_{43}e^{i\beta_4})e^{i\theta_{14}} & e^{i(\theta_{14}+\beta_4)} \\ -i(a_4 + s_{43}e^{-i\beta_4})e^{-i\theta_{14}} & e^{-i(\theta_{14}+\beta_4)} \end{bmatrix} \begin{bmatrix} \omega_{14} \\ \dot{s}_{43} \end{bmatrix} = \begin{bmatrix} ia_2e^{i\theta_{12}} \\ -ia_2e^{-i\theta_{12}} \end{bmatrix} \omega_{12}$$

$$\Delta = \begin{vmatrix} i(a_4 + s_{43}e^{i\beta_4})e^{i\theta_{14}} & e^{i(\theta_{14}+\beta_4)} \\ -i(a_4 + s_{43}e^{-i\beta_4})e^{-i\theta_{14}} & e^{-i(\theta_{14}+\beta_4)} \end{vmatrix} = ia_4e^{-i\beta_4} + is_{43} + ia_4e^{i\beta_4} + is_{43}$$

$$\Delta = ia_4(e^{i\beta_4} + e^{-i\beta_4}) + 2is_{43} = 2i(\cos\beta_4 + s_{43})$$

$$\omega_{14} = \frac{\omega_{12}}{\Delta} \begin{vmatrix} ia_2e^{i\theta_{12}} & e^{i(\theta_{14}+\beta_4)} \\ -ia_2e^{-i\theta_{12}} & e^{-i(\theta_{14}+\beta_4)} \end{vmatrix} = \frac{ia_2\omega_{12}}{\Delta}\left[e^{-i(\theta_{14}+\beta_4-\theta_{12})} + e^{i(\theta_{14}+\beta_4-\theta_{12})}\right]$$

$$\omega_{14} = \frac{2ia_2\omega_{12}\cos(\theta_{14}+\beta_4-\theta_{12})}{\Delta}$$

$$\omega_{14} = \frac{a_2\cos(\theta_{14}+\beta_4-\theta_{12})}{a_4\cos\beta_4 + s_{43}}\omega_{12}$$

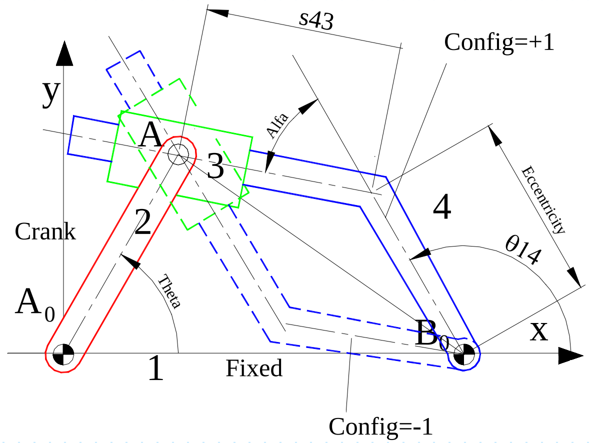
$$\dot{s}_{43} = \frac{\omega_{12}}{\Delta} \begin{vmatrix} i(a_4 + s_{43}e^{i\beta_4})e^{i\theta_{14}} & ia_2e^{i\theta_{12}} \\ -i(a_4 + s_{43}e^{-i\beta_4})e^{-i\theta_{14}} & -ia_2e^{-i\theta_{12}} \end{vmatrix}$$

$$\dot{s}_{43} = \frac{\omega_{12}}{\Delta}\left[a_4a_2e^{i(\theta_{14}-\theta_{12})} + a_2s_{43}e^{i(\theta_{14}+\beta_4-\theta_{12})} - a_4a_2e^{-i(\theta_{14}-\theta_{12})} - a_2s_{43}e^{-i(\theta_{14}+\beta_4-\theta_{12})}\right]$$

$$\dot{s}_{43} = \frac{a_2a_4\sin(\theta_{14}-\theta_{12}) + a_2s_{43}\sin(\theta_{14}+\beta_4-\theta_{12})}{a_4\cos\beta_4 + s_{43}}\omega_{12}$$

Acceleration Analysis

V



Acceleration Variables: $\alpha_{14},\ \ddot{s}_{43},\ \ \alpha_{12}$

$$i\alpha_{14}(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}-\omega_{14}^2(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}+i\omega_{14}\dot{s}_{43}e^{i(\theta_{14}+\beta_4)}+\ddot{s}_{43}e^{i(\theta_{14}+\beta_4)}+i\omega_{14}\dot{s}_{43}e^{i(\theta_{14}+\beta_4)}=ia_2\alpha_{12}e^{i\theta_{12}}-a_2\omega_{12}^2e^{i\theta_{12}}$$

or :

$$i\alpha_{14}(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}-\omega_{14}^2(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}+\ddot{s}_{43}e^{i(\theta_{14}+\beta_4)}+2i\omega_{14}\dot{s}_{43}e^{i(\theta_{14}+\beta_4)}=ia_2\alpha_{12}e^{i\theta_{12}}-a_2\omega_{12}^2e^{i\theta_{12}}$$

$$\vec{a}^{\,t}_{A_4}+\vec{a}^{\,n}_{A_4}+\vec{a}^{\,t}_{A_{3/4}}+\vec{a}^{\,c}_{A_{3/4}}=\vec{a}^{\,t}_{A_2}+\vec{a}^{\,n}_{A_2}$$

or

$$i\alpha_{14}(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}-\omega_{14}^2(a_4+s_{43}e^{i\beta_4})e^{i\theta_{14}}=ia_2\alpha_{12}e^{i\theta_{12}}-a_2\omega_{12}^2e^{i\theta_{12}}-\ddot{s}_{43}e^{i(\theta_{14}+\beta_4)}-2i\omega_{14}\dot{s}_{43}e^{i(\theta_{14}+\beta_4)}$$

$$\vec{a}^{\,t}_{A_4}+\vec{a}^{\,n}_{A_4}=\vec{a}^{\,t}_{A_2}+\vec{a}^{\,n}_{A_2}+\vec{a}^{\,t}_{A_{4/3}}+\vec{a}^{\,c}_{A_{4/3}}$$

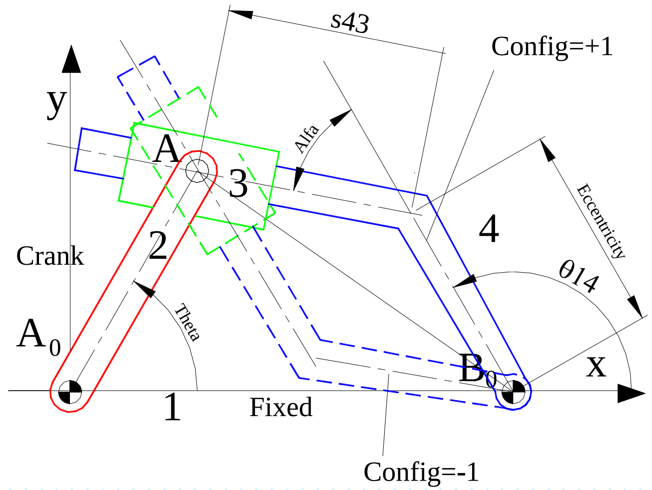
V

Unknown acceleration variables can be determined:

1. By drawing the vector polygon. (first draw the vectors whose both magnitude and direction are known. Leave two vectors whose direction are known but the magnitudes unknown on each side of the equation. Draw the vector directions. The intersection of these lines will define the solution.

2. By writing the acceleration loop equation and its complex conjugate and then solving two linear equations in 2 unknowns.

3. By Differentiating the equations obtained for velocity.



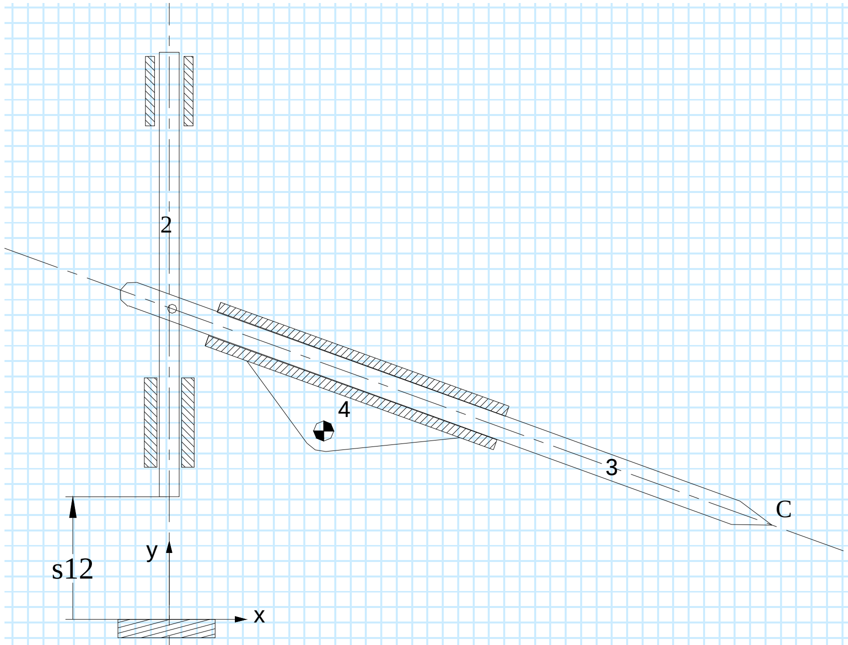
$$\omega_{14} = \frac{a_2 \cos(\theta_{14} + \beta_4 - \theta_{12})}{a_4 \cos\beta_4 + s_{43}} \omega_{12}$$

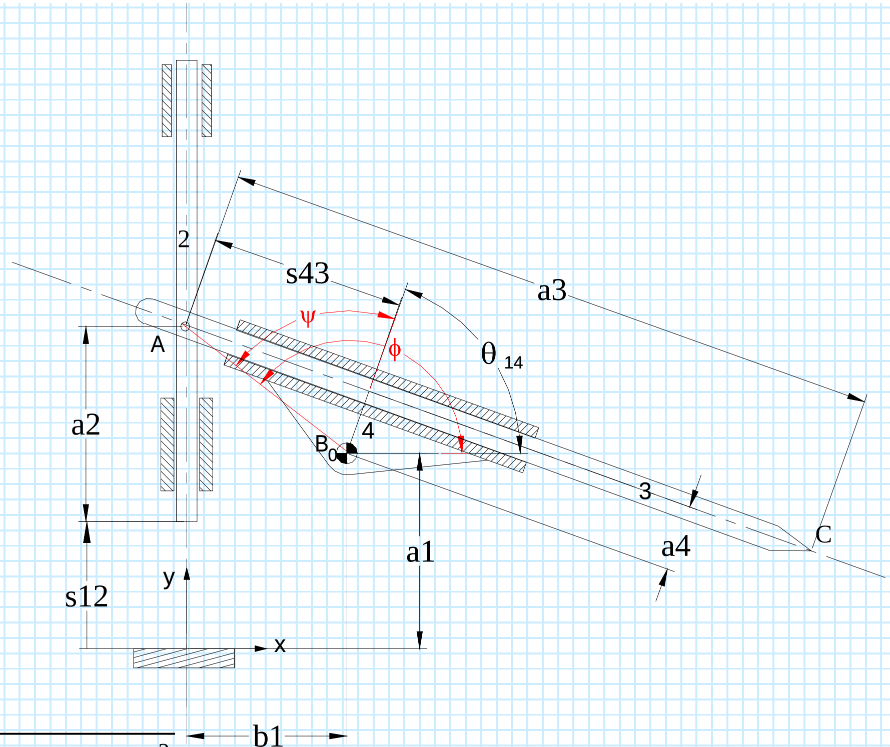
$$\alpha_{14} = \frac{a_2 \cos(\theta_{14} + \beta_4 - \theta_{12})}{a_4 \cos\beta_4 + s_{43}} \alpha_{12} + \frac{-a_2 \sin(\theta_{14} + \beta_4 - \theta_{12})(\omega_{14} - \omega_{12})(a_4 \cos\beta_4 + s_{43}) - \dot{s}_{43} a_2 \cos(\theta_{14} + \beta_4 - \theta_{12})}{(a_4 \cos\beta_4 + s_{43})^2} \omega_{12}$$

$$\dot{s}_{43} = \frac{a_2 a_4 \sin(\theta_{14} - \theta_{12}) + a_2 s_{43} \sin(\theta_{14} + \beta_4 - \theta_{12})}{a_4 \cos\beta_4 + s_{43}} \omega_{12}$$

$$\ddot{s}_{43} = \frac{a_2 a_4 \sin(\theta_{14} - \theta_{12}) + a_2 s_{43} \sin(\theta_{14} + \beta_4 - \theta_{12})}{a_4 \cos\beta_4 + s_{43}} \alpha_{12}$$

$$+ \frac{\left[a_2 a_4 \cos(\theta_{14} - \theta_{12})(\omega_{14} - \omega_{12}) + a_2 s_{43} \cos(\theta_{14} + \beta_4 - \theta_{12})(\omega_{14} - \omega_{12}) + a_2 \dot{s}_{43} \sin(\theta_{14} + \beta_4 - \theta_{12})\right](a_4 \cos\beta_4 + s_{43}) - \dot{s}_{43}\left[a_2 a_4 \sin(\theta_{14} - \theta_{12}) + a_2 s_{43} \sin(\theta_{14} + \beta_4 - \theta_{12})\right]}{\left(a_4 \cos\beta_4 + s_{43}\right)^2}$$





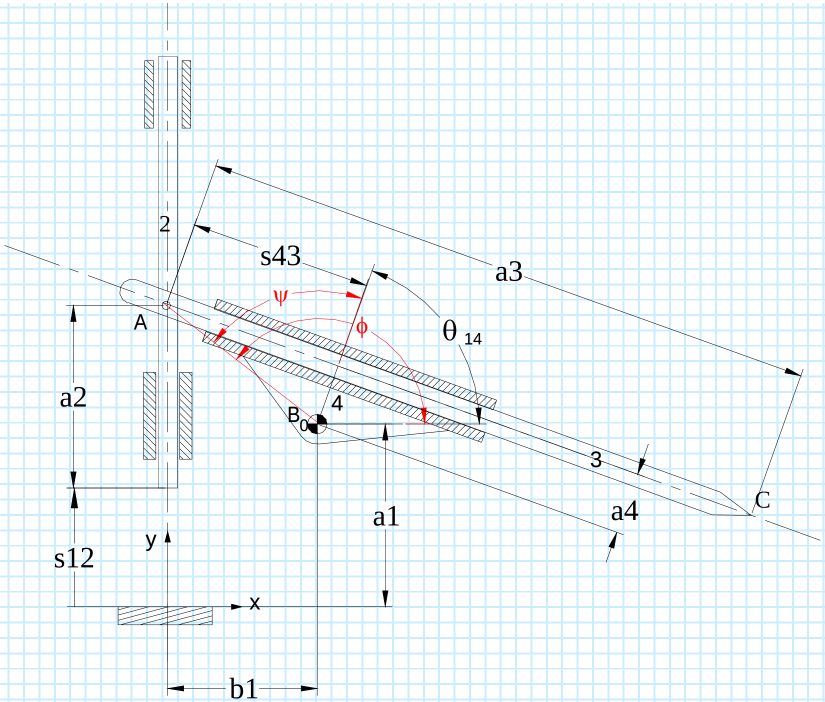
Position Analysis

1. $B_0A = s = \sqrt{b_1^{\,2} + (s_{12} + a_2 - a_1)^2}$
2. $\phi = \tan^{-1}\left[-b_1, (s_{12} + a_2 - a_1)\right]$
3. $s_{43} = \sqrt{s^2 - a_4^{\,2}}$
4. $\psi = \cos^{-1}\left[\dfrac{a_4}{s}\right]$
5. $\theta_{14} = \phi - \psi$

$\vec{r}_C = i(s_{12} + a_2) + a_3 e^{i(\theta_{14} - \pi/2)}$

$\vec{r}_C = i(s_{12} + a_2) - i a_3 e^{i\theta_{14}}$

6. $x_C = +a_3 \sin\theta_{14}$
7. $y_C = s_{12} + a_2 - a_3 \cos\theta_{14}$



Loop closure Equation:

$$i(s_{12}+a_2)=b_1+ia_1+a_4e^{i\theta_{14}}+s_{43}e^{i(\theta_{14}+\pi/2)}$$

or

$$i(s_{12}+a_2-a_1)=b_1+a_4e^{i\theta_{14}}+is_{43}e^{i\theta_{14}}$$

Velocity loop Equation:

$$i\dot{s}_{12}=+ia_4\omega_{14}e^{i\theta_{14}}-s_{43}\omega_{14}e^{i\theta_{14}}+i\dot{s}_{43}e^{i\theta_{14}}$$

or

$$i\dot{s}_{12}=+i\omega_{14}e^{i\theta_{14}}(a_4+is_{43})+i\dot{s}_{43}e^{i\theta_{14}}$$

$$\vec{V}_{A3}=\vec{V}_{A2}=\vec{V}_{A4}+\vec{V}_{A3/4}$$

Solving the velocity loop equation and its complex conjugate for ω_{14} and $\dot{s}_{43}$:

$$\Delta=\begin{vmatrix} i(a_4+is_{43})e^{i\theta_{14}} & ie^{i\theta_{14}} \\ -i(a_4-is_{43})e^{-i\theta_{14}} & -ie^{-i\theta_{14}} \end{vmatrix}=2is_{43}$$

$$\omega_{14}=\frac{\dot{s}_{12}}{\Delta}\begin{vmatrix} i & ie^{i\theta_{14}} \\ -i & -ie^{-i\theta_{14}} \end{vmatrix}=\frac{\dot{s}_{12}}{2is_{43}}\left[e^{-i\theta_{14}}-e^{i\theta_{14}}\right]=-\frac{\sin(\theta_{14})}{s_{43}}\dot{s}_{12}$$

$$\dot{s}_{43}=\frac{\dot{s}_{12}}{\Delta}\begin{vmatrix} i(a_4+is_{43})e^{i\theta_{14}} & i \\ -i(a_4-is_{43})e^{-i\theta_{14}} & -i \end{vmatrix}=\frac{\dot{s}_{12}}{2is_{43}}\left[(a_4+is_{43})e^{i\theta_{14}}-(a_4-is_{43})e^{-i\theta_{14}}\right]=\frac{\dot{s}_{12}}{2is_{43}}\left[a_4(e^{i\theta_{14}}-e^{-i\theta_{14}})+is_{43}(e^{i\theta_{14}}+e^{-i\theta_{14}})\right]$$

$$\dot{s}_{43}=\frac{\dot{s}_{12}}{2is_{43}}\left[2ia_4\sin(\theta_{14})+2is_{43}\cos(\theta_{14})\right]=\left[\frac{a_4}{s_{43}}\sin(\theta_{14})+\cos(\theta_{14})\right]\dot{s}_{12}$$

Acceleration loop Equation:

$$i\ddot{s}_{12} = +ia_4\alpha_{14}e^{i\theta_{14}} - a_4\omega_{14}^2 e^{i\theta_{14}} - s_{43}\alpha_{14}e^{i\theta_{14}} - is_{43}\omega_{14}^2 e^{i\theta_{14}} - \dot{s}_{43}\omega_{14}e^{i\theta_{14}} + i\ddot{s}_{43}e^{i\theta_{14}} - \dot{s}_{43}\omega_{14}e^{i\theta_{14}}$$

or

$$i\ddot{s}_{12} = i\alpha_{14}e^{i\theta_{14}}(a_4 + is_{43}) - \omega_{14}^2 e^{i\theta_{14}}(a_4 + is_{43}) - 2\dot{s}_{43}\omega_{14}e^{i\theta_{14}} + i\ddot{s}_{43}e^{i\theta_{14}}$$

$$\vec{a}_{A3} = \vec{a}_{A2} = \quad \vec{a}_4^t \quad + \quad \vec{a}_4^n \quad + \quad \vec{a}_{A3/4}^c \quad + \quad \vec{a}_{A3/4}^t$$

Solving the acceleration loop equation and its complex conjugate for α_{14} and $\ddot{s}_{43}$:

$$\Delta = \begin{vmatrix} i(a_4 + is_{43})e^{i\theta_{14}} & ie^{i\theta_{14}} \\ -i(a_4 - is_{43})e^{-i\theta_{14}} & -ie^{-i\theta_{14}} \end{vmatrix} = 2is_{43}$$

Note that the discriminant D is the same as in the velocity analysis

$$\alpha_{14} = \frac{1}{\Delta}\begin{vmatrix} \left[i\ddot{s}_{12} + \omega_{14}^2 e^{i\theta_{14}}(a_4 + is_{43}) + 2\dot{s}_{43}\omega_{14}e^{i\theta_{14}}\right] & ie^{i\theta_{14}} \\ \left[-i\ddot{s}_{12} + \omega_{14}^2 e^{-i\theta_{14}}(a_4 - is_{43}) + 2\dot{s}_{43}\omega_{14}e^{-i\theta_{14}}\right] & -ie^{-i\theta_{14}} \end{vmatrix} = \frac{\ddot{s}_{12}}{2is_{43}}\left[e^{-i\theta_{14}} - e^{i\theta_{14}}\right] + \frac{2i\omega_{14}^2 a_4}{2is_{43}} - \frac{4i\dot{s}_{43}\omega_{14}}{2is_{43}}$$

$$\alpha_{14} = -\frac{\sin\theta_{14}}{s_{43}}\,\ddot{s}_{12} + \frac{a_4}{s_{43}}\omega_{14}^2 - \frac{2\dot{s}_{43}\omega_{14}}{s_{43}}$$

$$\dot{s}_{43} = \frac{1}{\Delta}\begin{vmatrix} i(a_4 + is_{43})e^{i\theta_{14}} & \left[i\ddot{s}_{12} + \omega_{14}^2 e^{i\theta_{14}}(a_4 + is_{43}) + 2\dot{s}_{43}\omega_{14}e^{i\theta_{14}}\right] \\ -i(a_4 - is_{43})e^{-i\theta_{14}} & \left[-i\ddot{s}_{12} + \omega_{14}^2 e^{-i\theta_{14}}(a_4 - is_{43}) + 2\dot{s}_{43}\omega_{14}e^{-i\theta_{14}}\right] \end{vmatrix}$$

$$\dot{s}_{43} = \frac{\ddot{s}_{12}}{2is_{43}}\left[a_4(e^{i\theta_{14}} - e^{-i\theta_{14}}) + is_{43}(e^{i\theta_{14}} - e^{-i\theta_{14}})\right] + \frac{\omega_{14}^2}{2is_{43}}2i(a_4^2 + s_{43}^2) + \frac{4ia_4\dot{s}_{43}\omega_{14}}{2is_{43}}$$

$$\dot{s}_{43} = \frac{\ddot{s}_{12}}{s_{43}}\left[a_4\sin\theta_{14} + s_{43}\cos\theta_{14}\right] + \frac{\omega_{14}^2}{s_{43}}(a_4^2 + s_{43}^2) + \frac{2a_4\dot{s}_{43}\omega_{14}}{s_{43}}$$

Or, differentiating the velocities:

$$\omega_{14} = -\frac{\sin\theta_{14}}{s_{43}}\dot{s}_{12}$$

$$\alpha_{14} = \frac{d\omega_{14}}{dt} = -\frac{\sin\theta_{14}}{s_{43}}\ddot{s}_{12} - \frac{\dot{s}_{12}}{s_{34}^{2}}\left[-s_{43}\omega_{14}\cos\theta_{14} - \dot{s}_{43}\sin\theta_{14}\right]$$

$$\dot{s}_{43} = \left[\frac{a_4}{s_{43}}\sin\theta_{14} + \cos\theta_{14}\right]\dot{s}_{12}$$

$$\ddot{s}_{43} = \left[\frac{a_4}{s_{43}}\sin\theta_{14} + \cos\theta_{14}\right]\ddot{s}_{12} + \dot{s}_{12}\left[\frac{a_4}{s_{43}^{2}}(s_{43}\omega_{14}\cos\theta_{14} - \dot{s}_{43}\sin\theta_{14}) - \omega_{14}\sin\theta_{14}\right]$$

Position, Velocity and Acceleration of point C

$$\vec{r}_C = i(s_{12} + a_2) + a_3 e^{i(\theta_{14} - \pi/2)} = i(s_{12} + a_2) - i a_3 e^{i\theta_{14}}$$

$$\vec{v}_C = i\dot{s}_{12} + a_3\omega_{14}e^{i\theta_{14}}$$

$$\mathrm{v}_{\mathrm{Cx}} = a_3\omega_{14}\cos\theta_{14}$$

$$\mathrm{v}_{\mathrm{Cy}} = \dot{s}_{12} + a_3\omega_{14}\sin\theta_{14}$$

$$\vec{a}_C = i\ddot{s}_{12} + a_3\alpha_{14}e^{i\theta_{14}} + i a_3\omega_{14}^{2}e^{i\theta_{14}}$$

$$\mathrm{a}_{\mathrm{Cx}} = a_3\alpha_{14}\cos\theta_{14} - a_3\omega_{14}^{2}\sin\theta_{14}$$

$$\mathrm{a}_{\mathrm{Cy}} = \ddot{s}_{12} + a_3\alpha_{14}\sin\theta_{14} + a_3\omega_{14}^{2}\cos\theta_{14}$$