Force Analysis

Principles of Dynamics:

Newton's Laws of motion. :

1. ***A body will remain in a state of rest, or of uniform motion in a straight line unless it is acted by external forces to change its state.***

2. ***The rate of change of momentum of a body acted upon by an external Force (or forces) is proportional to the resultant external force F and in the direction of that force:***

$$\vec{F} = \frac{d(m\vec{v})}{dt}$$

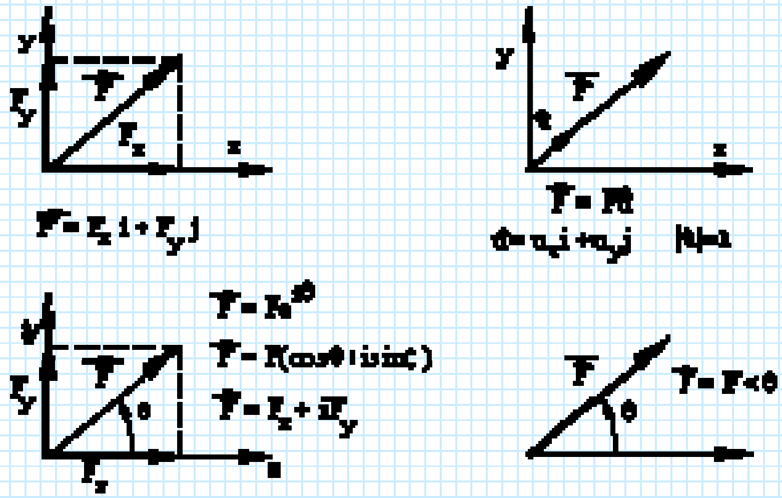
Where m is the mass and **v** is the velocity of the body. For constant mass; m:

F = m **a**

Where **a** is the acceleration of the body.

3. ***To every action of a force there is an equal and opposite reaction***.

Force is a vectorial quantity

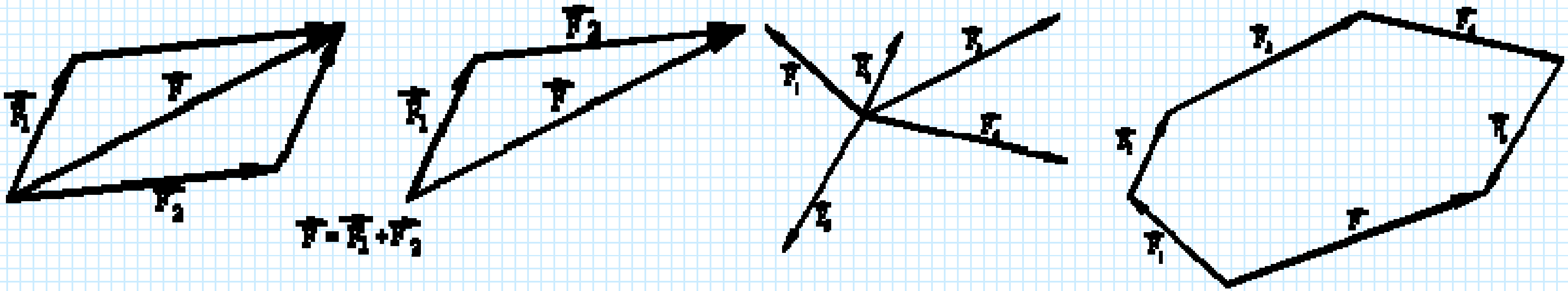


For several concurrent forces

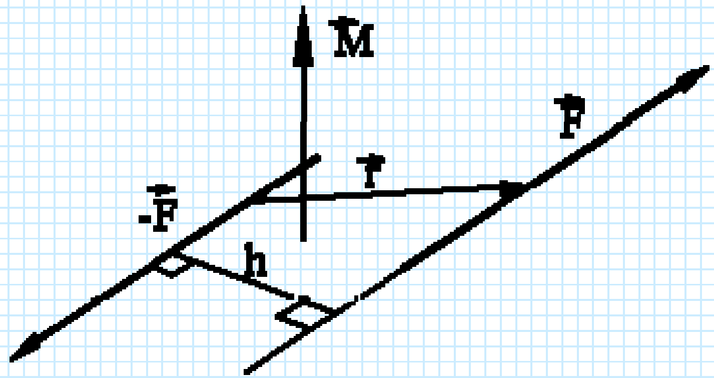
$$\vec{F} = \sum_i \vec{F}_i$$

$$F_x = \sum_i F_{ix} \qquad F_y = \sum_i F_{iy} \qquad F_z = \sum_i F_{iz}$$

Paralelogram law of addition



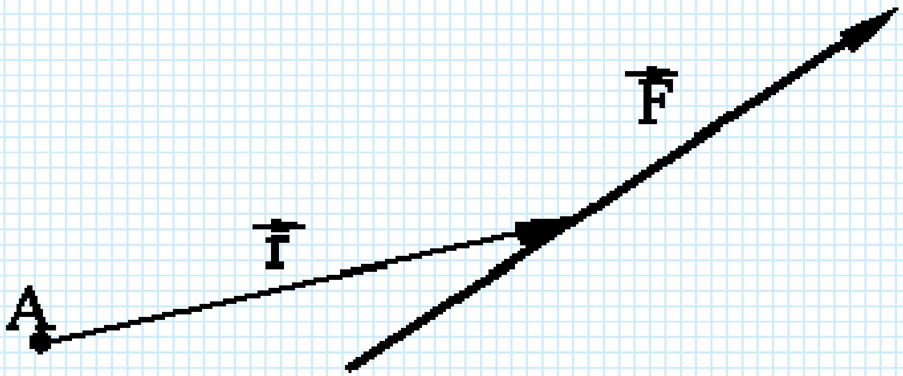
couple



a) The moment vector does not have a point of application. Hence, it is a free vector.
b) The relative position vector, **r**, is in between any two points on the lines of action of the forces forming the couple.
c) The force couple that creates a moment **M** is not unique, e.g. there are other force couples that may create the same moment.

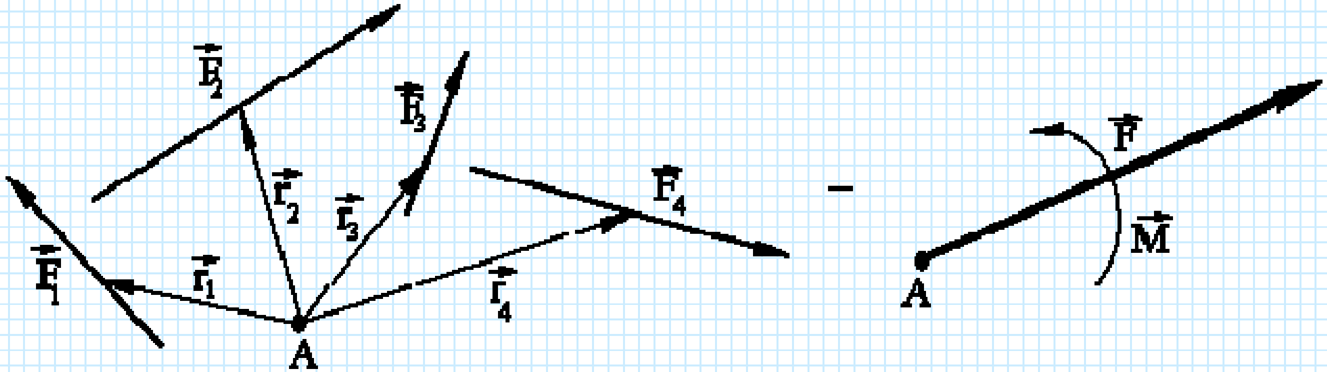
Moment of a force

M = **r** x **F**

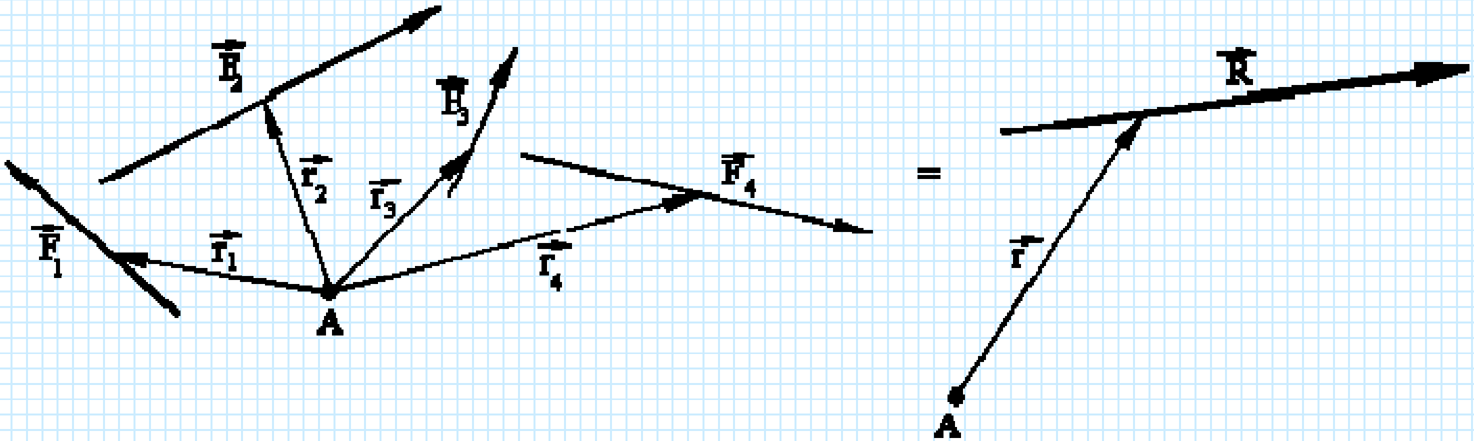


Resultant of nonconcurrent forces

$$\vec{F} = \sum_i \vec{F}_i \qquad \vec{M} = \sum_i (\vec{r}_i \, x \, \vec{F}_i)$$



or

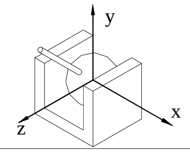
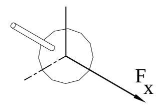
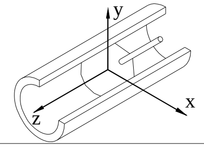
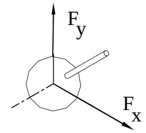
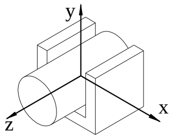
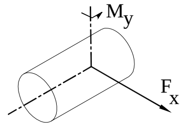
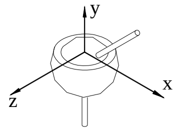
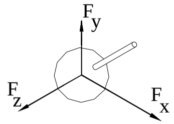
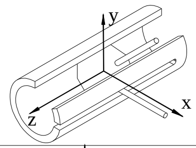
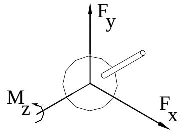
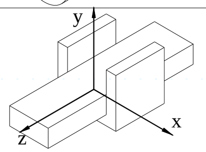
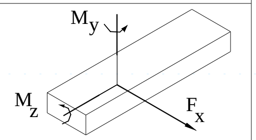


Forces in Machine Systems

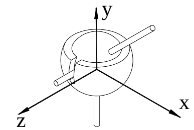
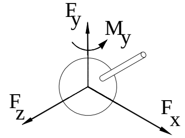
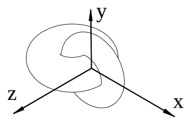
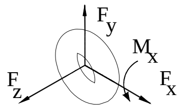
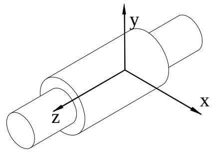
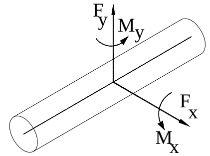
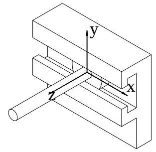
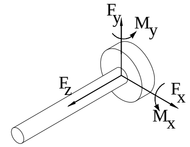
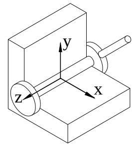
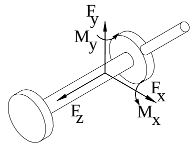
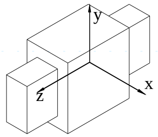
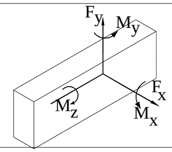
Reaction Forces: are commonly called the ***joint forces*** in machine systems since the action and reaction between the bodies involved will be through the contacting kinematic elements of the links that form a joint. **The joint forces are along the direction for which the degree-of-freedom is restricted.**

$$F_{ij} = - F_{jj}$$

REACTION FORCES AT KINEMATIC PAIRS (JOINT FORCES)

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	JOINT SHAPE	JOINT FORCE
5	3	2	Sphere between parallel planes		
4	3	1	Sphere in a cylinder		
	2	2	Cylinder between parallel planes		
3	3	0	Spherical pair (Ball joint)		
	2	1	Slotted sphere in a cylinder		
	1	2	Plane joint		

REACTION FORCES AT KINEMATIC PAIRS (JOINT FORCES)

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	JOINT SHAPE	JOINT FORCE
2	2	0	Slotted sphere		
	2	0	Torus		
	1	1	Cylindrical joint		
	1	1	Slotted cylinder		
1	1	0	Revolute pair (turning joint)		
	0	1	Prismatic pair (sliding joint)		

Forces in Machine Systems

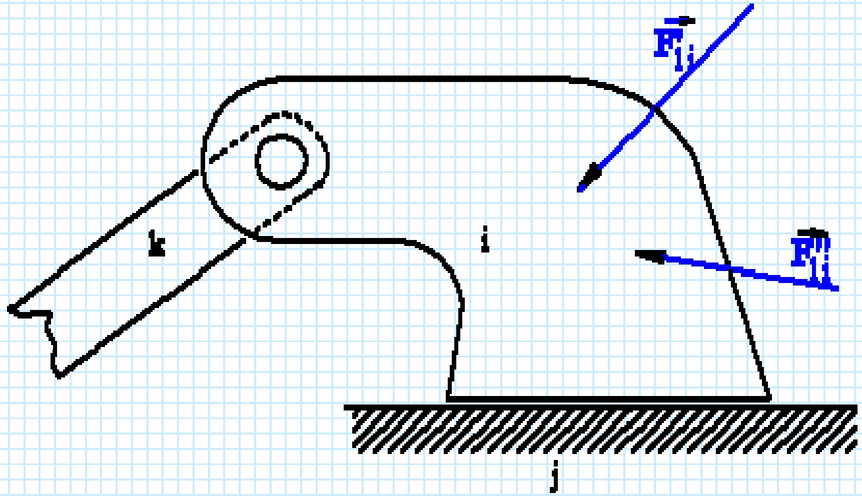
a) Joint Forces

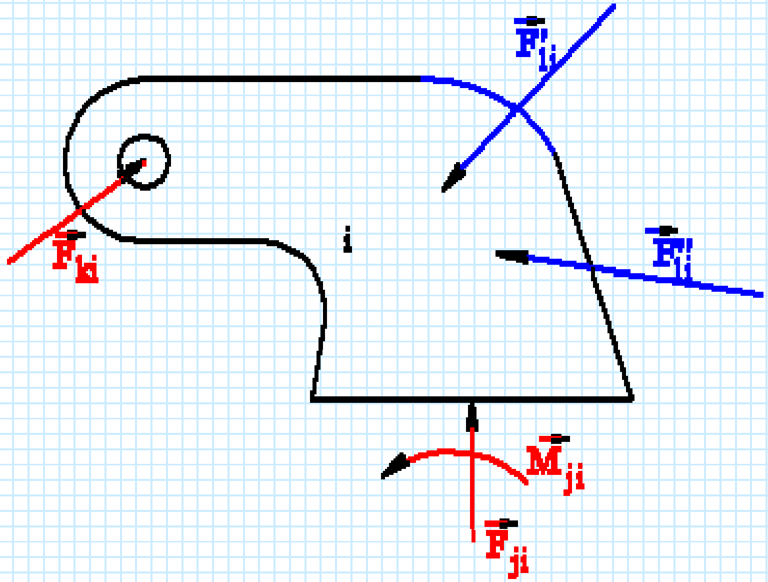
b) Physical Forces

***c) Friction or Resisting Force*:**

***d) Inertial Forces*.**

Free-Body Diagram





Static Equilibrium

$$\sum \vec{\mathbf{F}} = 0 \qquad \sum \vec{\mathbf{M}} = 0$$

In space, these two vector equations yield six scalar equations:

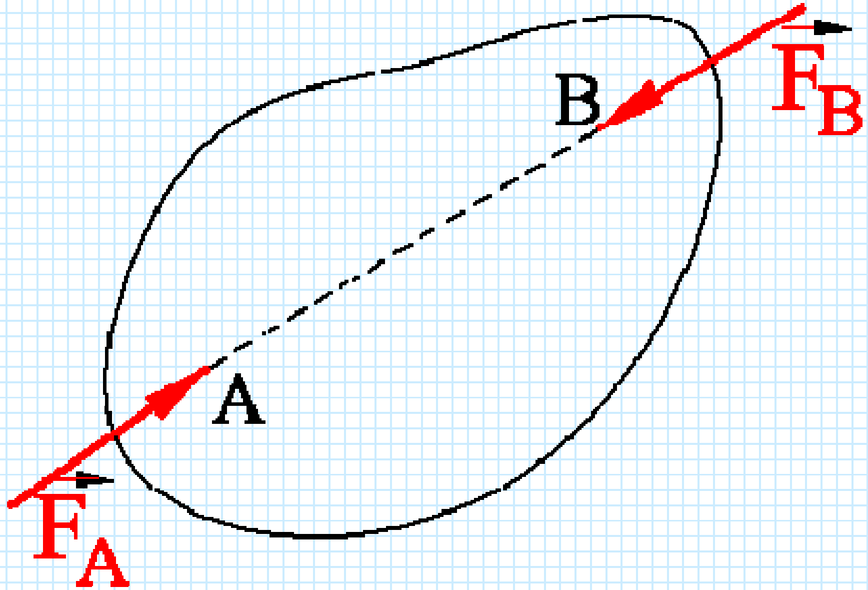
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

In case of coplanar force systems, there are three scalar equations:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

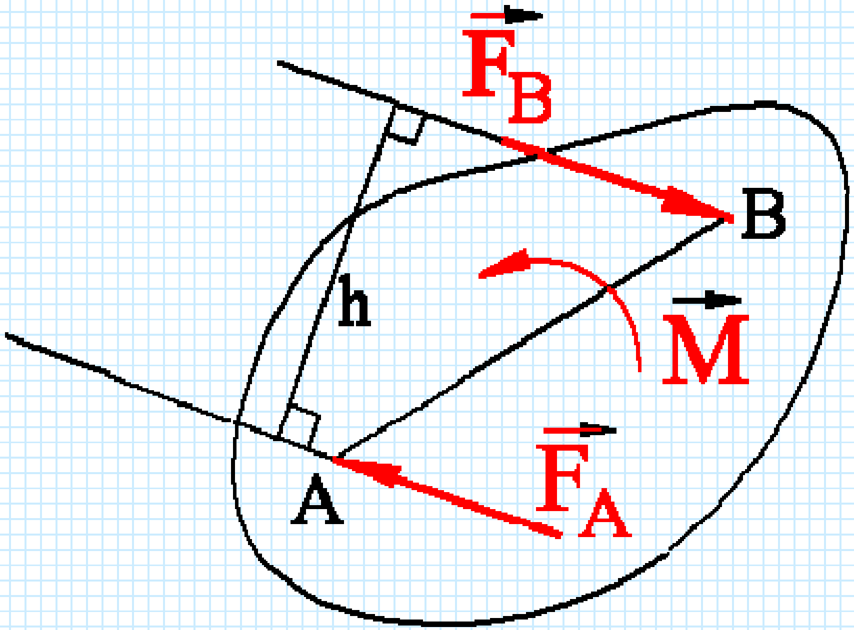
Two Force Member



$$\vec{F}_A = -\vec{F}_B$$

In a Two-Force member, the forces must be equal and opposite and must have the same line of action

Two force and one moment member:

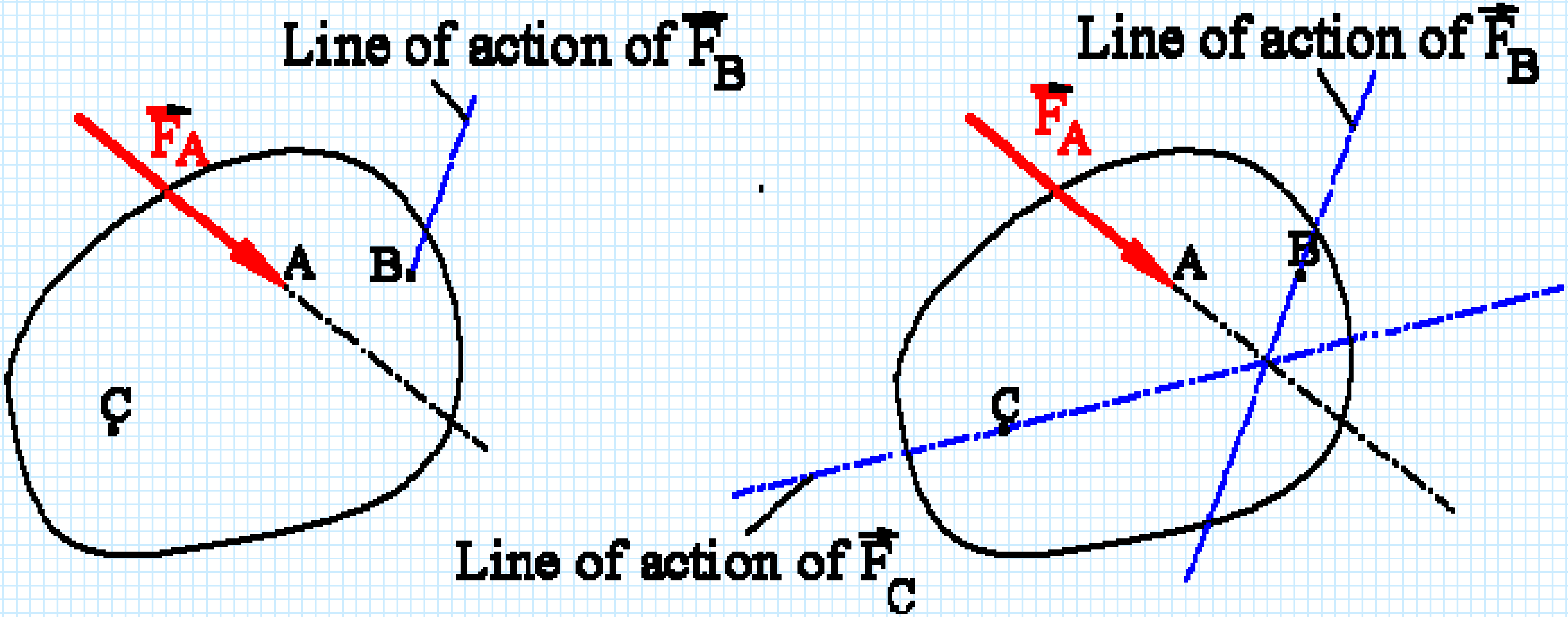


$$\vec{F}_A = \vec{F}_B$$

$$M = hF_A$$

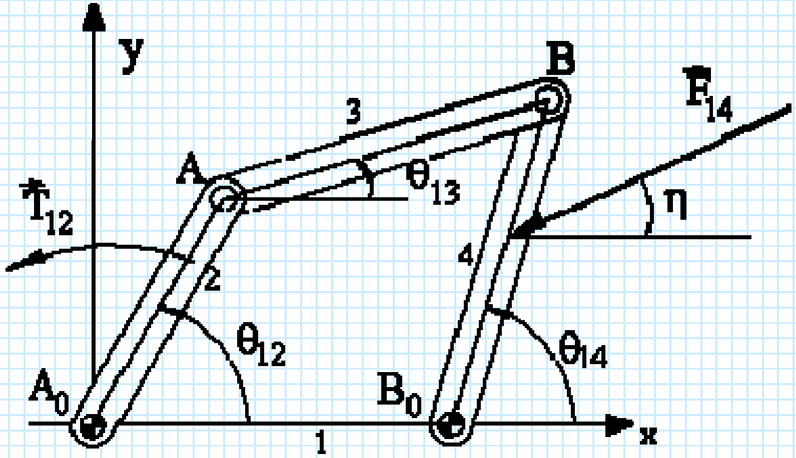
Two forces form a couple whose moment is equal in magnitude but in opposite sense to the applied moment

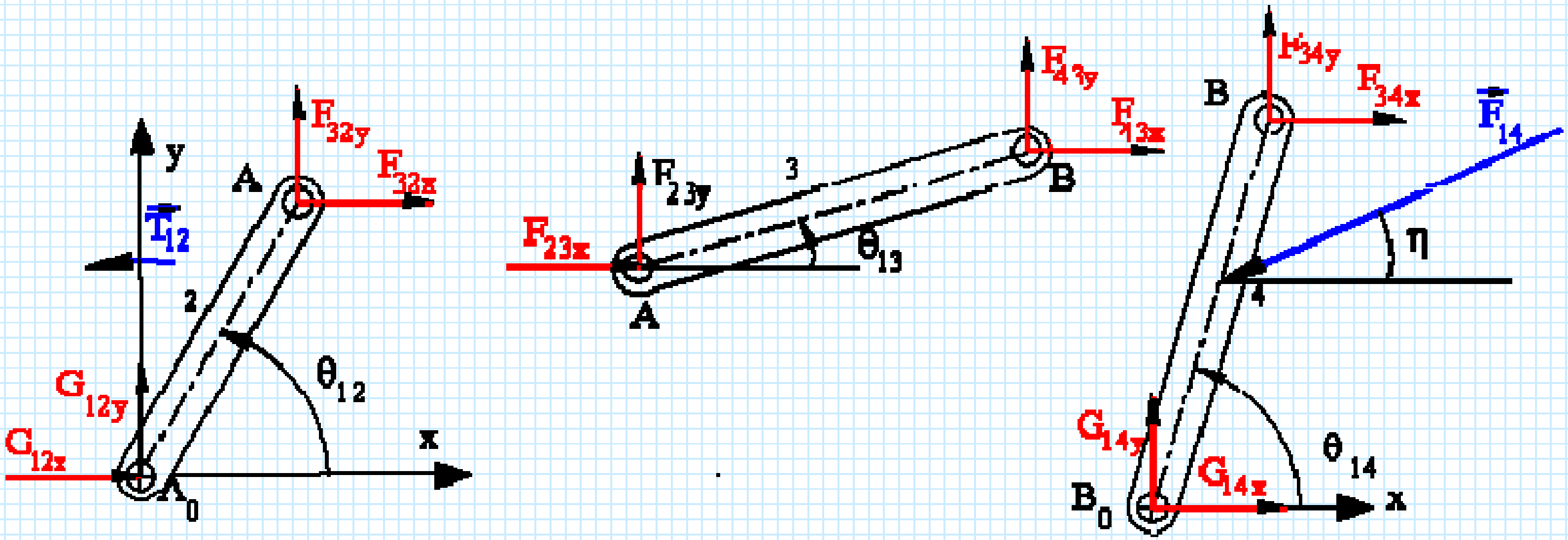
Three Force Member

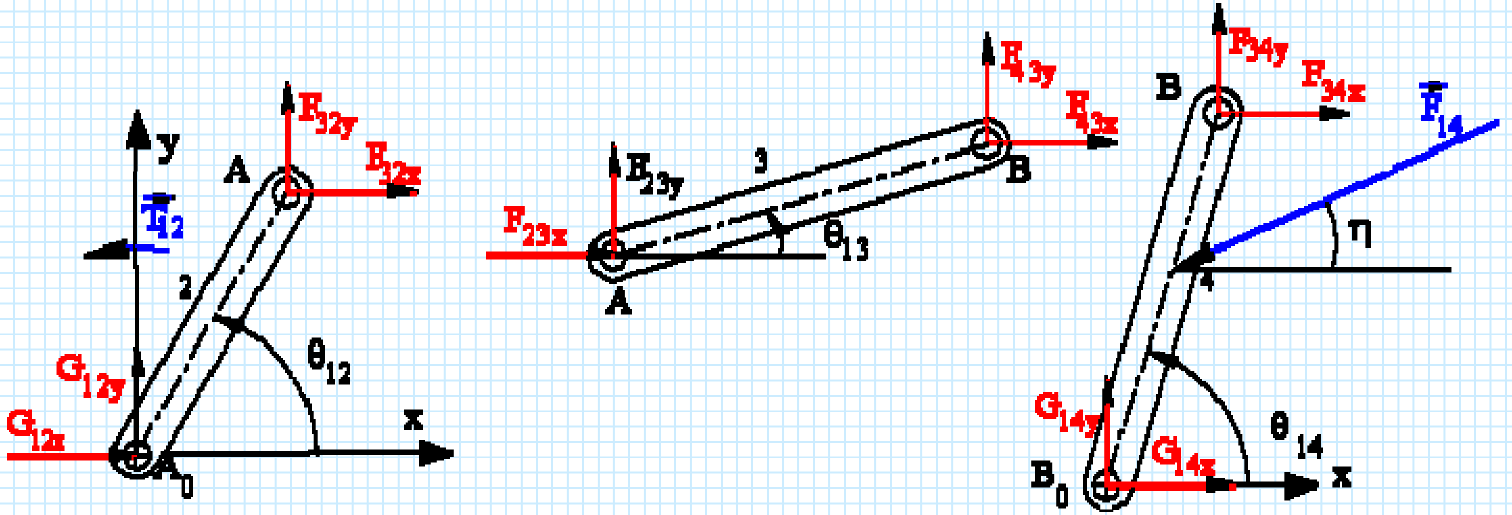


STATIC FORCE ANALYSIS OF MACHINERY

Systems without Resisting Force







Link 2

$F_{32y} + G_{12y} = 0 \quad (\Sigma F_y = 0)$

$F_{32x} + G_{12x} = 0 \quad (\Sigma F_x = 0)$

$F_{32y}\, a_2 \cos\theta_{12} - F_{32x}\, a_2 \sin\theta_{12} + T_{12} = 0 \quad (\Sigma M_{Ao} = 0)$

For link 3:

$F_{23x} + F_{43x} = 0 \quad (\Sigma F_x = 0)$

$F_{23y} + F_{43y} = 0 \quad (\Sigma F_y = 0)$

$F_{43x}\, a_3 \sin\theta_{13} + F_{43y}\, a_3 \cos\theta_{13} = 0 \quad (\Sigma M_A = 0)$

For link 4:

$F_{34x} + G_{14x} - F_{14} \cos\eta = 0 \quad (\Sigma F_x = 0)$

$F_{34y} + G_{14y} - F_{14} \sin\eta = 0 \quad (\Sigma F_y = 0)$

$F_{34x}\, a_4 \sin\theta_{14} + F_{34y}\, a_4 \cos\theta_{14} + F_{14}\, r_4 (\cos\eta \sin\theta_{14} - \sin\eta \cos\theta_{14}) = 0 \quad (\Sigma M_{Bo} = 0)$

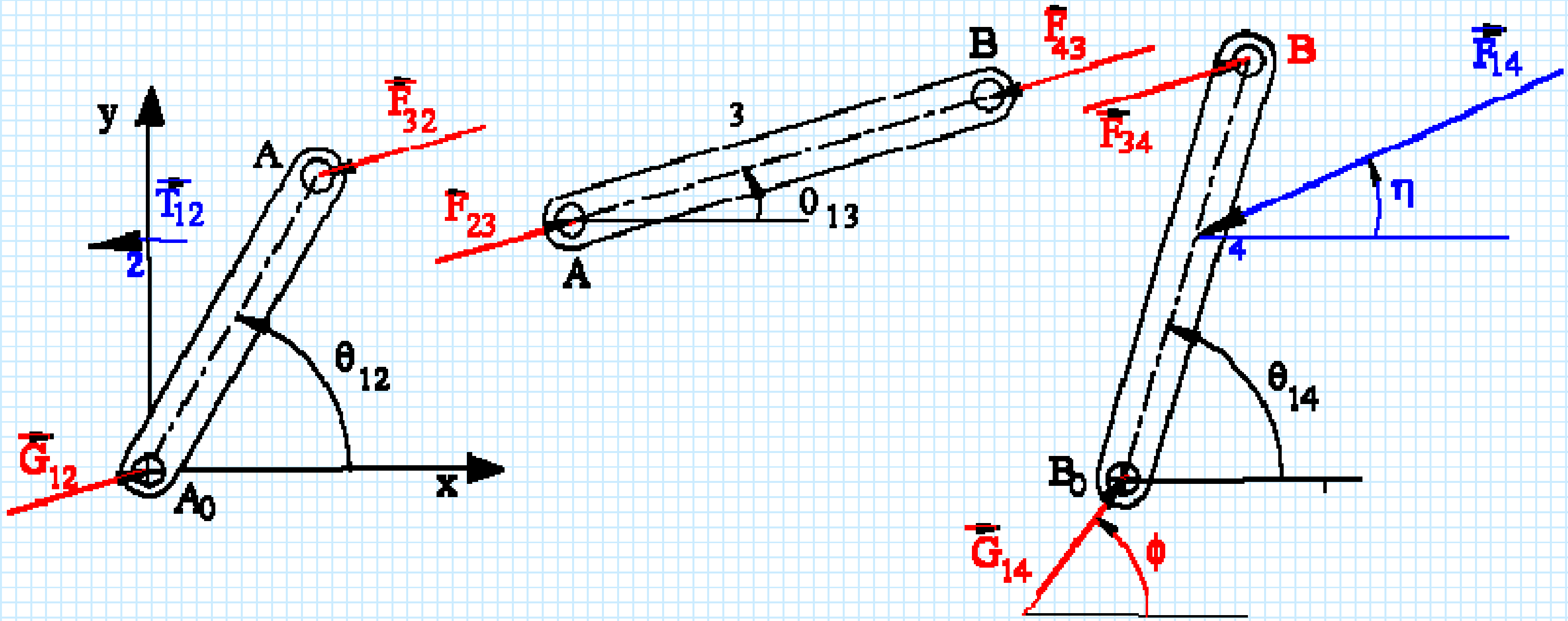
Due to Third Law:

$F_{32y} = - F_{23y}$

$F_{32x} = - F_{23x}$

$F_{43y} = - F_{34y}$

$F_{43x} = - F_{34x}$



$\mathbf{F}_{23}$=F ∠ θ_{13}

$$\mathbf{F}_{43} = -\mathbf{F}_{32} = -\mathbf{F}_{34} = -\mathbf{G}_{12} = -\mathbf{F}_{23}$$

$$- F_{14}\cos\eta + F_{34}\cos\theta_{13} + G_{14}\cos\phi = 0 \quad \text{or} \quad -F_{14}\cos\eta + F_{34}\cos\theta_{13} + G_{14x} = 0$$

$$- F_{14}\sin\eta + F_{34}\sin\theta_{13} + G_{14}\sin\phi = 0 \quad \text{or} \quad - F_{14}\sin\eta + F_{34}\sin\theta_{13} + G_{14y} = 0$$

$$r_4F_{14}(\cos\eta\ \sin\theta_{14} - \sin\eta\ \cos\theta_{14}) + a_4F_{34}(\sin\theta_{13}\ \cos\theta_{14} - \cos\theta_{13}\ \sin\theta_{14}) = 0$$

And

$$a_2F_{32}(\cos\theta_{13}\sin\theta_{12} - \sin\theta_{13}\ \cos\theta_{12}) + T_{12} = 0$$

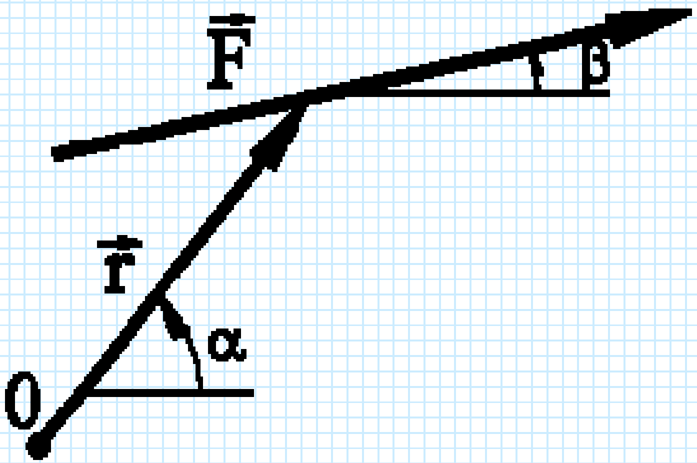
$$(F_{32} = - F_{34})$$

In general:

a) For two-force members you don't have to write the equilibrium equations. You can simply state that the forces are equal and opposite and their line of action coincides with the line joining the points of applications.

b) For two-force plus a moment members you must write the moment equilibrium equation only. The two forces are equal and opposite and they form a couple equal and opposite to the moment applied.

c) In case of three or four force members, the three equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$) must be written.



$\mathbf{F} = F\mathbf{v}$ $\qquad$ $\mathbf{r} = r\mathbf{u}$

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = r\mathbf{u}\times F\mathbf{v} = rF(\mathbf{u}\times\mathbf{v})$$

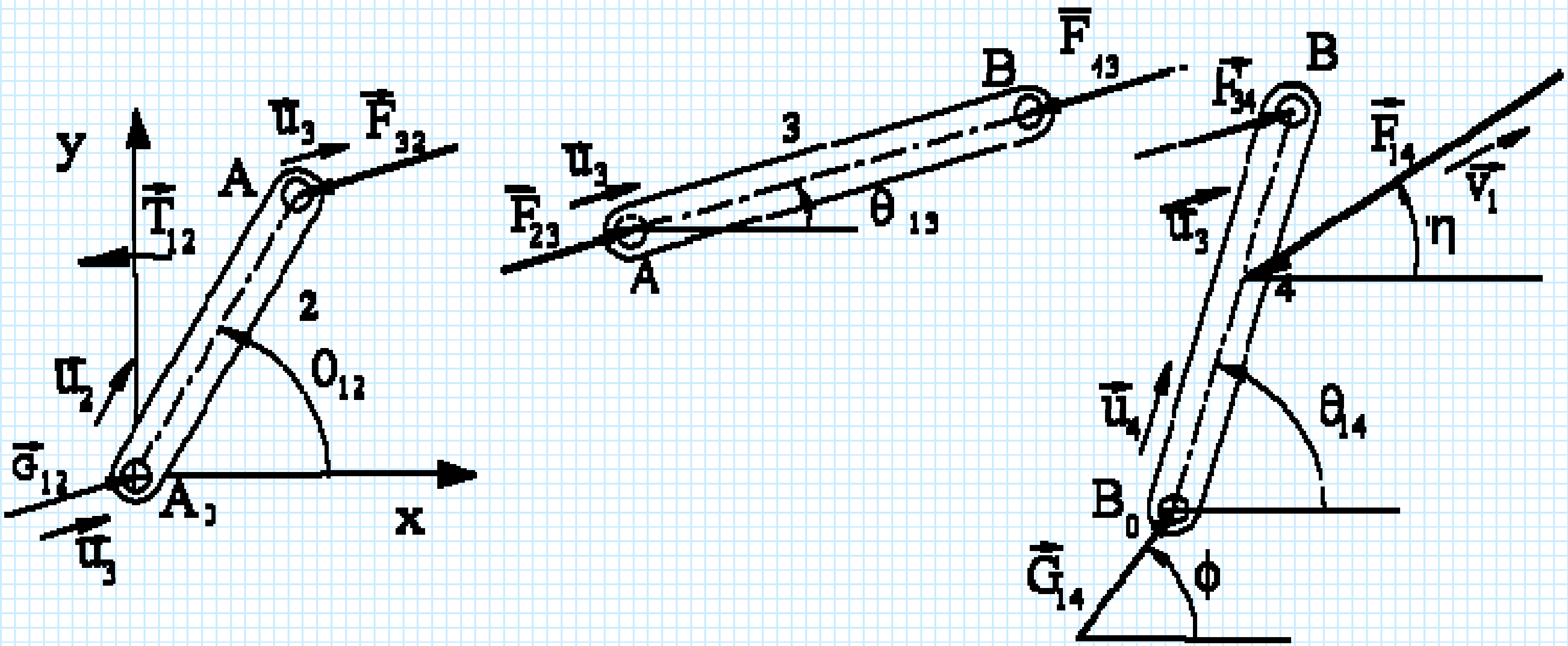
$$\mathbf{u} = 1 \angle \alpha = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$$

$$\mathbf{v} = 1 \angle \beta = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$$

$$\mathbf{u} \times \mathbf{v} = (\sin\beta\cos\alpha - \cos\beta\sin\alpha)\ \mathbf{k}$$

Or: $\qquad \mathbf{u} \times \mathbf{v} = \sin(\beta - \alpha)\,\mathbf{k}$

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = r\,F\sin(\beta - \alpha)\,\mathbf{k}$$



$a_4\mathbf{u}_4 \times F_{34}\mathbf{u}_3 + r_4\mathbf{u}_4 \times (-F14)\mathbf{v}_1$ $(\Sigma \mathbf{M}_{Bo} = 0)$

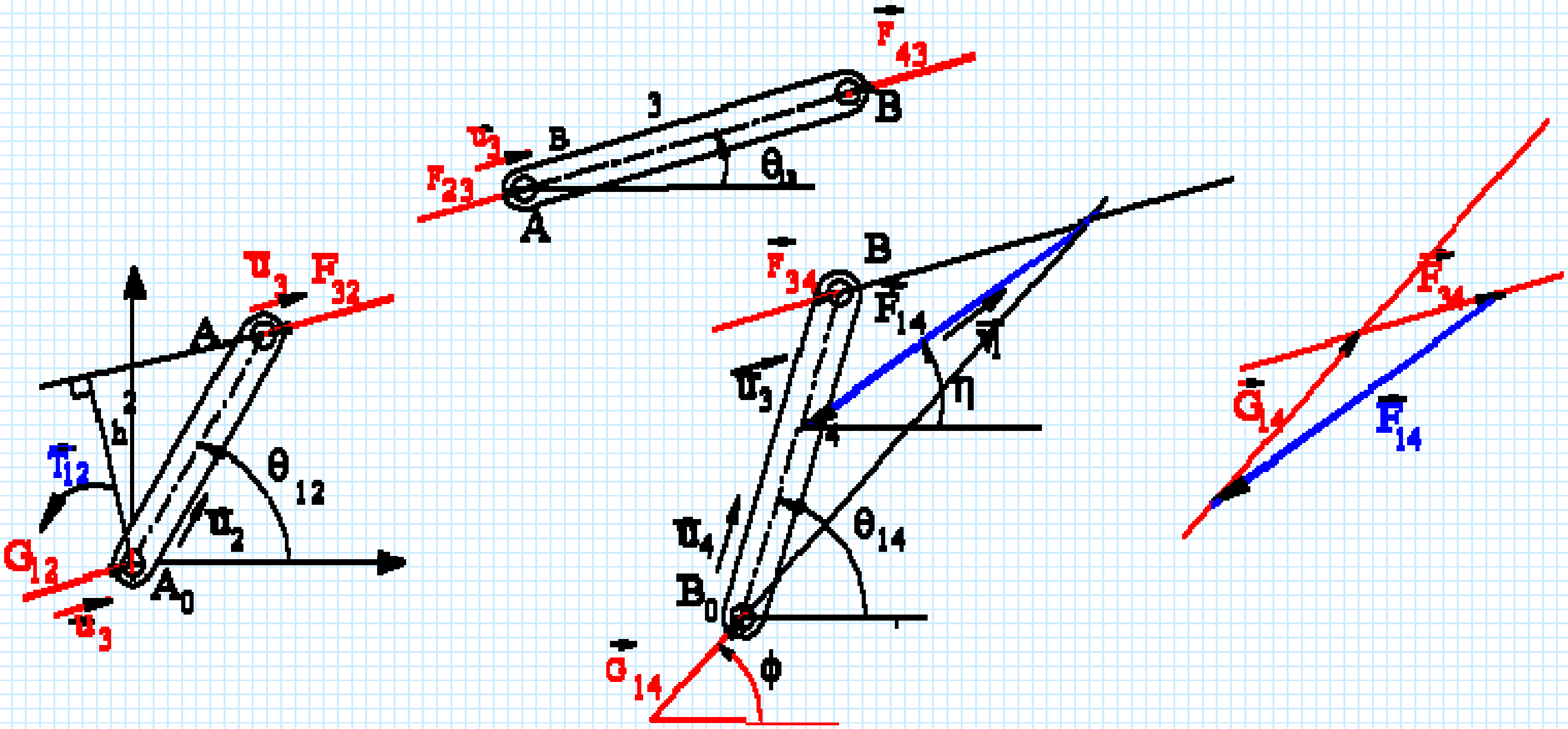
$a_2\mathbf{u}_2 \times (-F_{32})\mathbf{u}_3 + T_{12}\,\mathbf{k} = 0$ $(\Sigma \mathbf{M}_{Ao} = 0)$

Which result in:

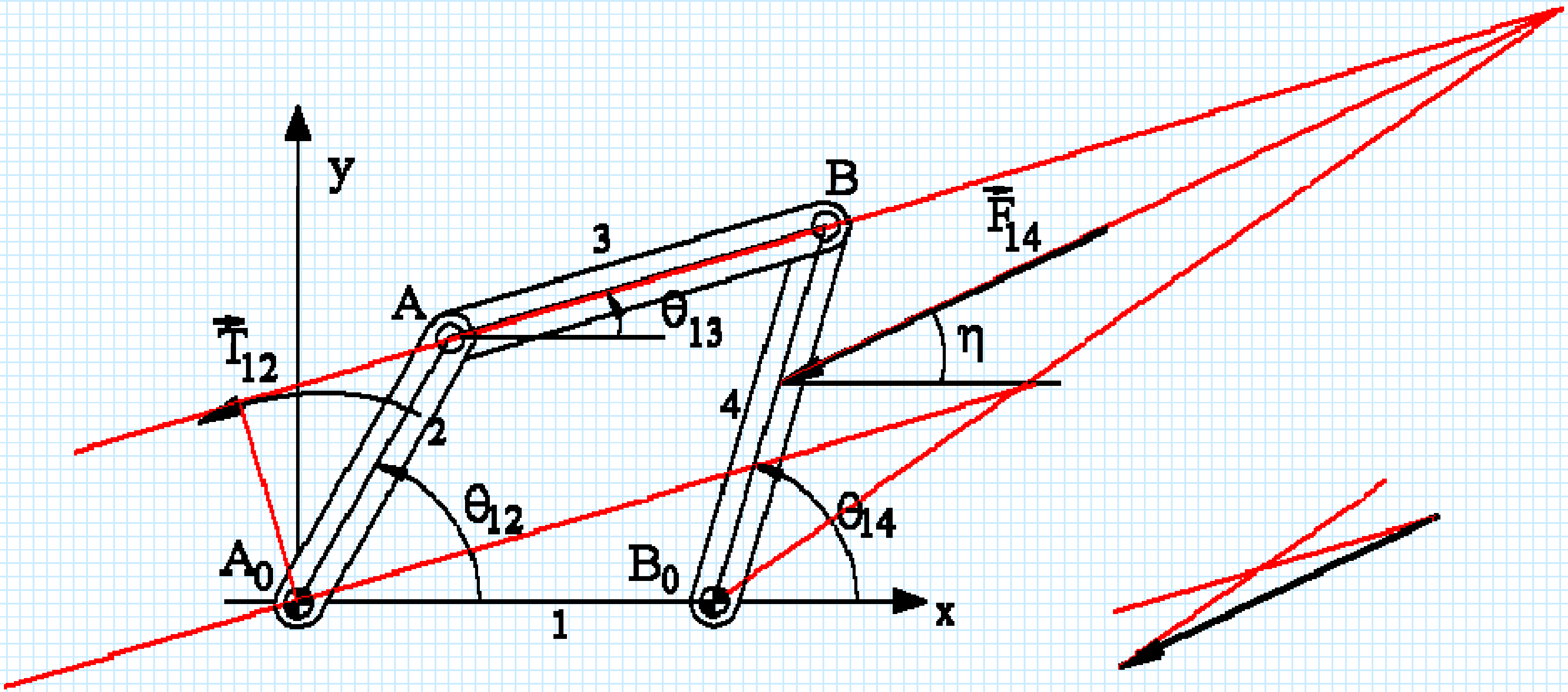
$a_4 F_{34} \sin(\theta_{13}-\theta_{14}) - r_4 F_{14} \sin(\eta-\theta_{14}) = 0$

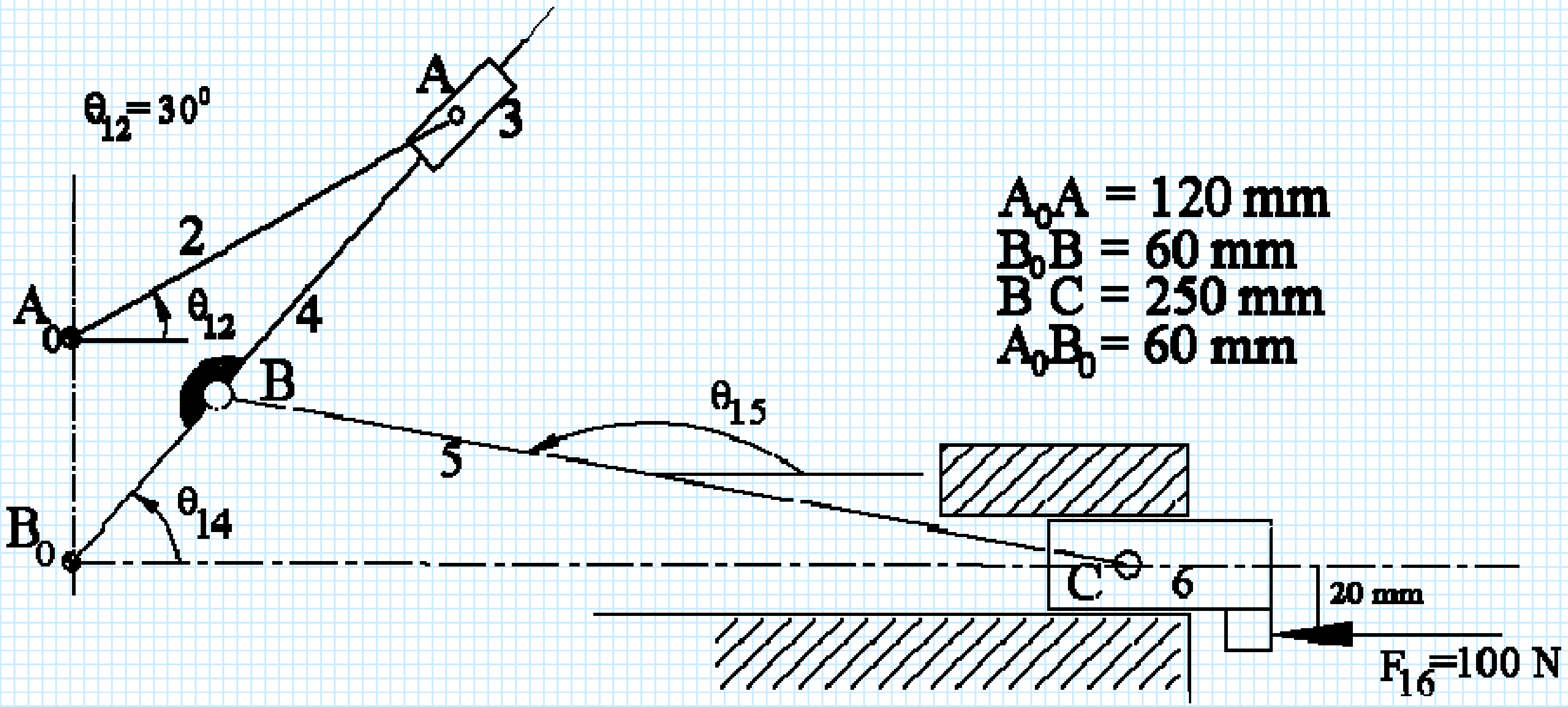
$-a_2 F_{32} \sin(\theta_{13}-\theta_{12}) + T_{12} = 0$

Graphical Solution

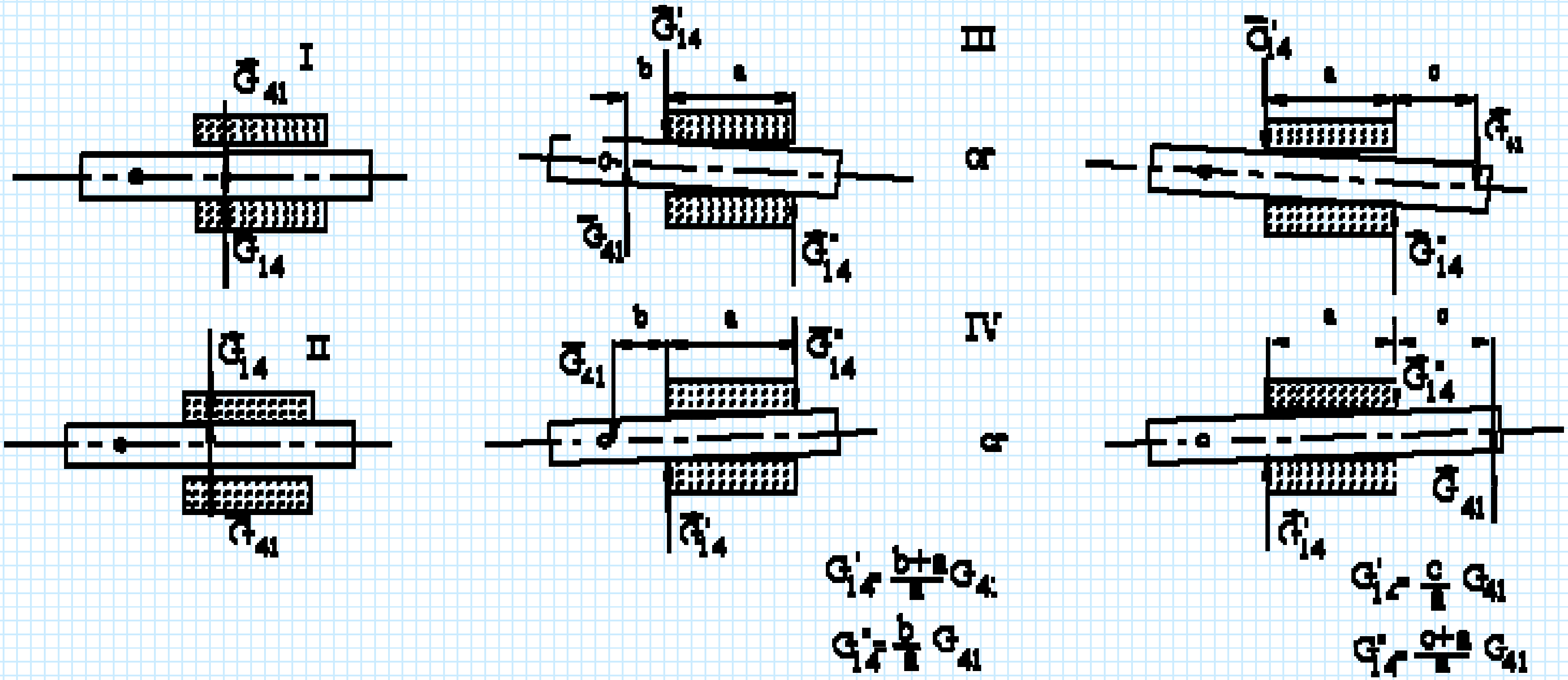


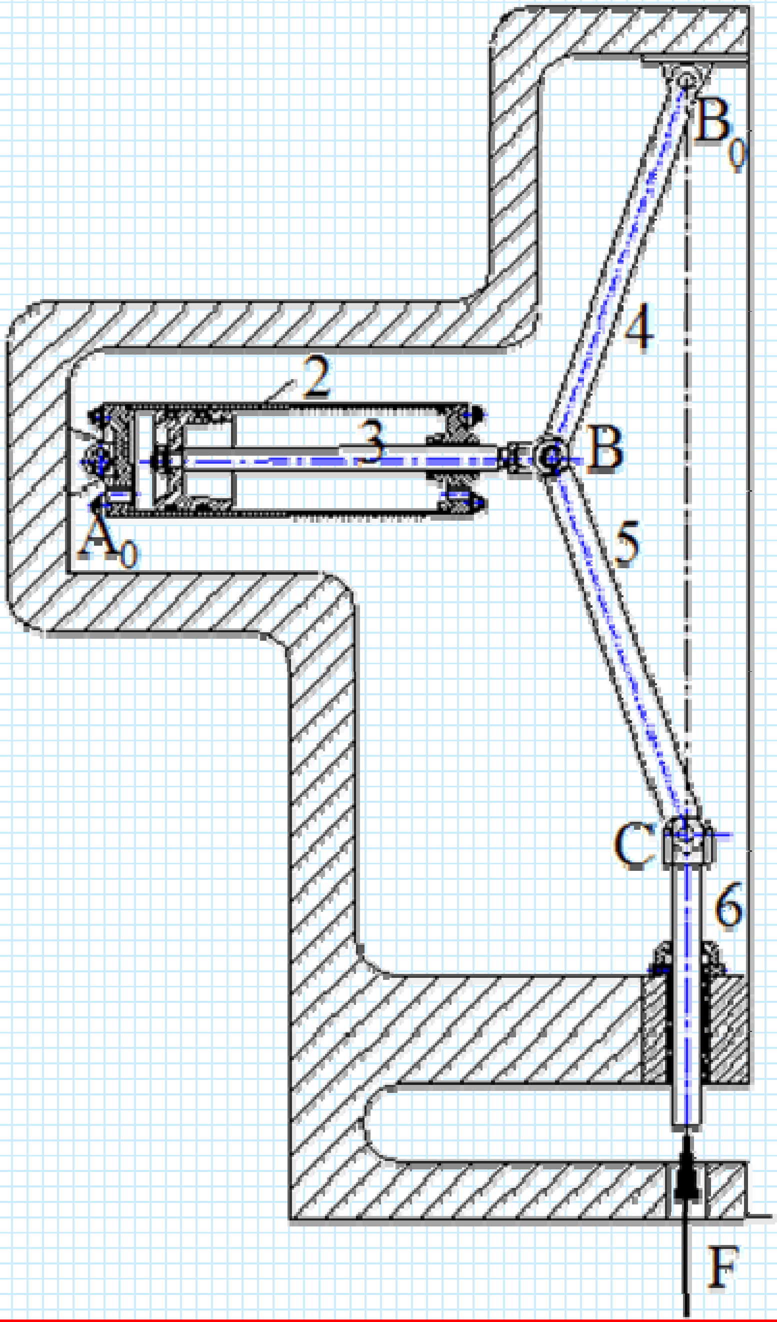
Graphical Solution



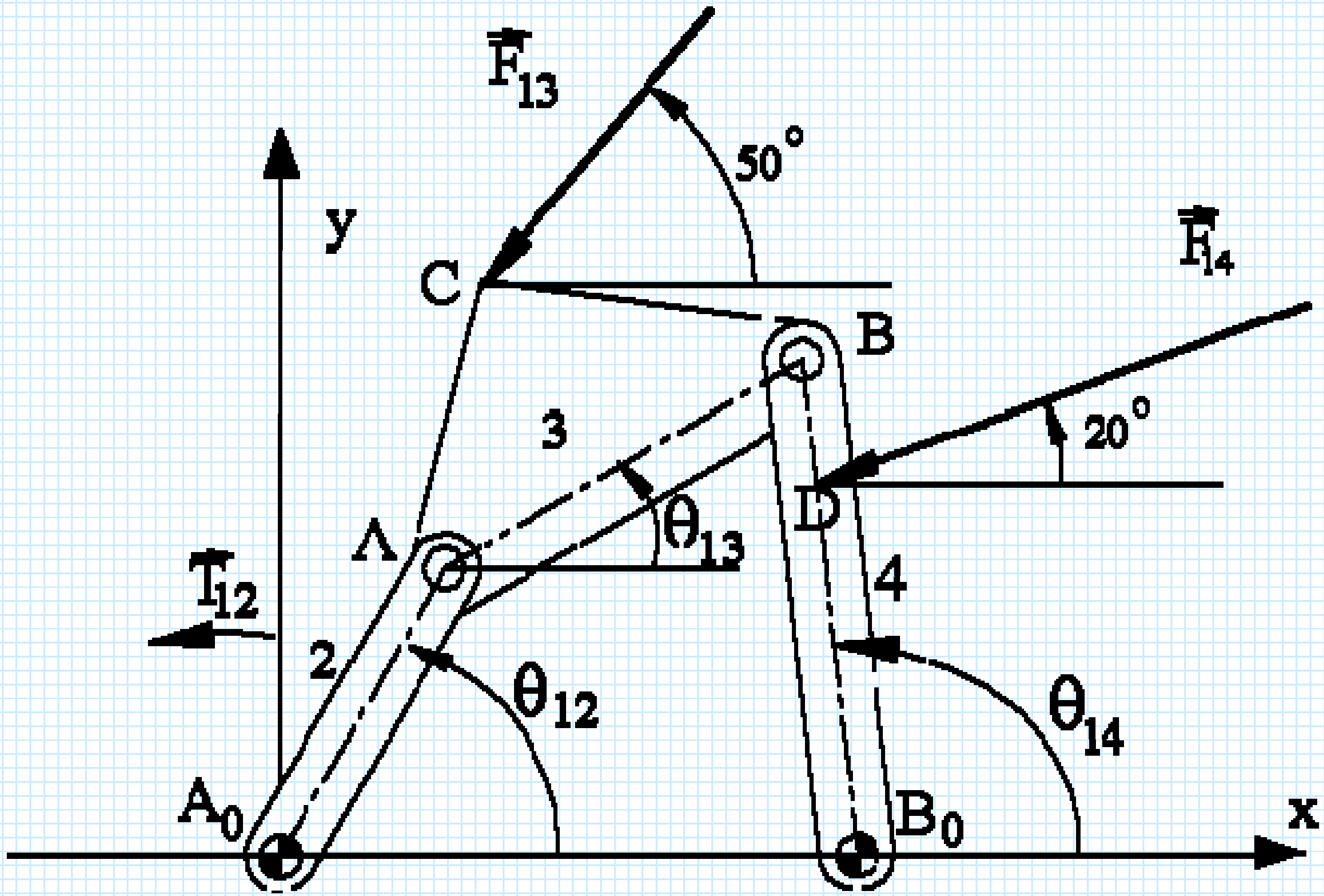

θ12= 30°
A
3
2
A0
θ12
4
B
θ15
5
B0
θ14
C
6
20 mm
F16=100 N
A0A = 120 mm
B0B = 60 mm
B C = 250 mm
A0B0= 60 mm

Modes of Contact in Prismatic Joints




B0
4
2
3
B
A0
5
C
6
F

Principle of Superposition



Systems with Resisting Force

1. **Static Frictional Force** :

$$R_{32} = - \mu F_{32}$$

μ, is known as the coefficient of static friction.

2. **Sliding Frictional Force**

$$R_{32} = - \mu F_{32},$$

μ is the coefficient of sliding friction, which is less than the coefficient of static friction. Sliding friction is also known as Coulomb friction.

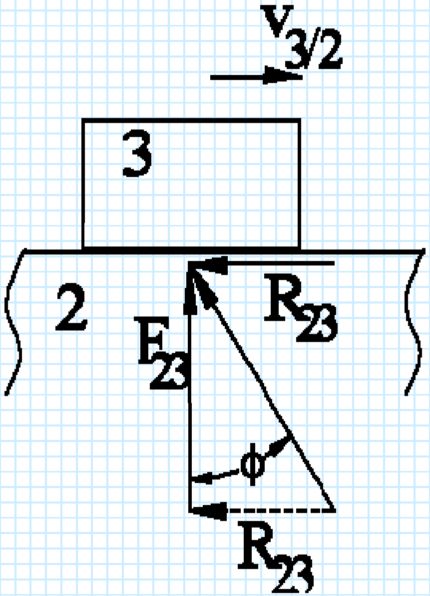
– Experiments have shown that the static or sliding friction force does not depend on the area of contact. It depends on the types of materials in contact, on the surface quality in contact and the type of film formed between the contacting surfaces. The coefficient of static friction is slightly larger then the coefficient of sliding friction.

3. **Viscous Damping Force**

$$R_{32} = -cv_{2/3}$$

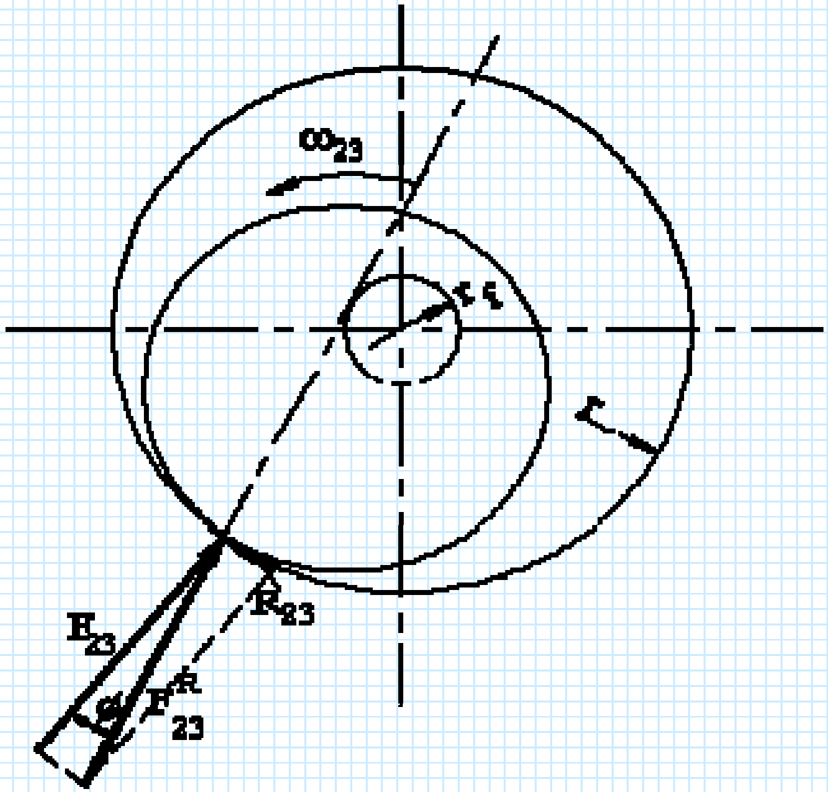
Where c is the coefficient of viscous friction and $v_{3/2}$ is the relative velocity. Viscous friction assumes that there is a fluid film between the two surfaces in contact.

friction angle :



$$\tan\phi = \mu = \frac{R_{32}}{F_{32}}$$

friction circle.

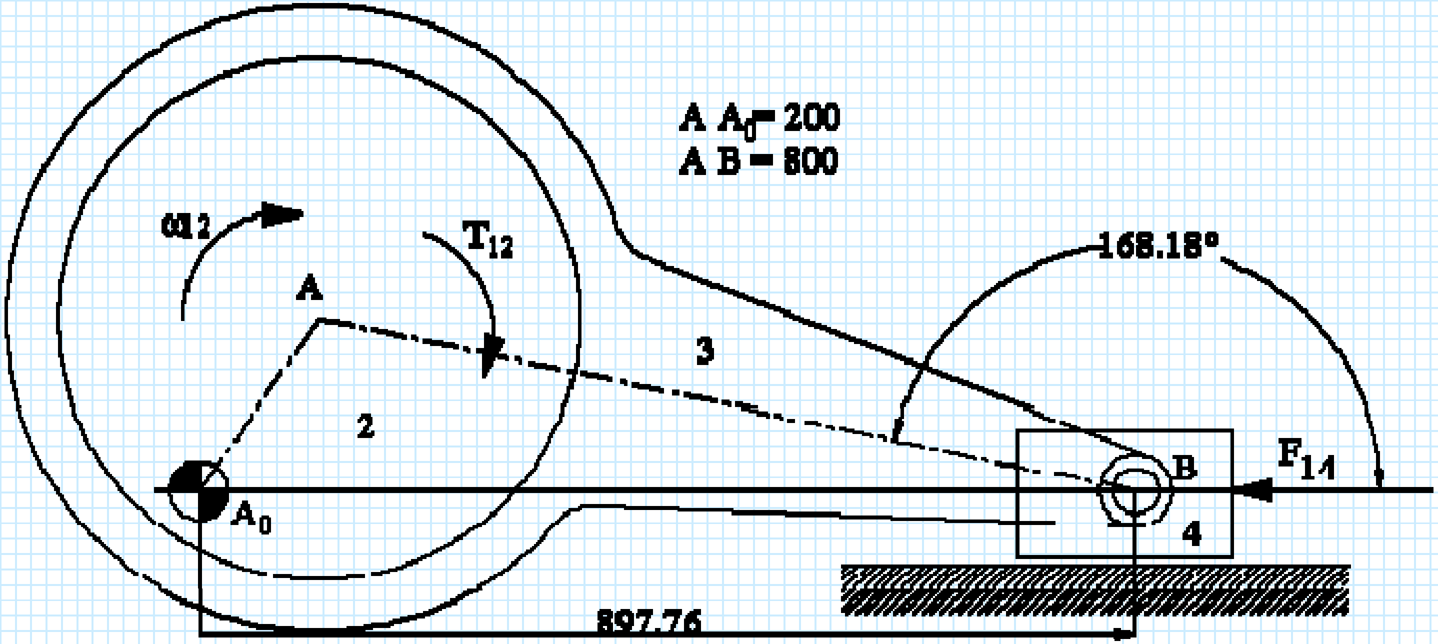


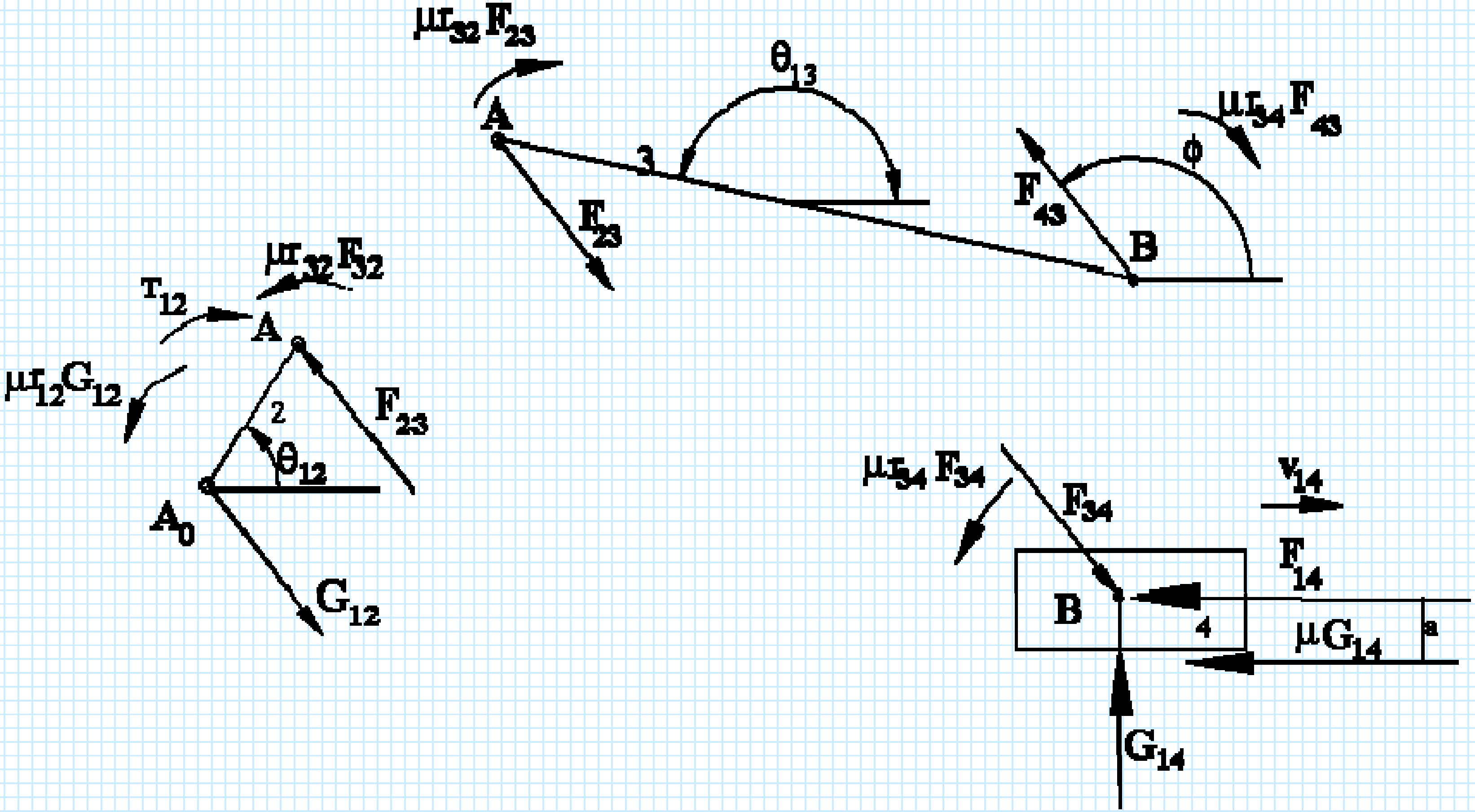
$$\vec{F}^{R}_{23} = \vec{R}_{23} + \vec{F}_{23}$$

$$r_f = r\sin\phi \cong r\tan\phi = \mu r$$

$$\mathbf{M}_{32} = \mathbf{r_f}\,\mathbf{F}^{\mathbf{R}}_{32}$$ Friction Torque

$\theta_{12}=55^0$

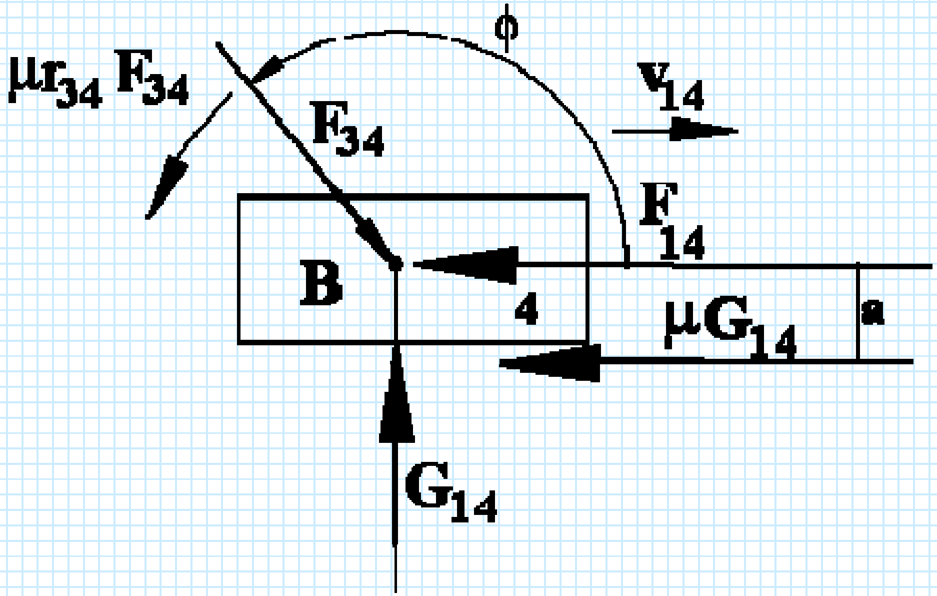



μr32 F23
θ13
A
3
F23
μr34 F43
φ
F43
B
μr32 F32
T12
A
μr12 G12
2
F23
θ12
A0
G12
μr34 F34
F34
V14
F14
B
4
μG14
a
G14

$-F_{34}\cos(\phi)-\mu G_{14}-F_{14}=0$ ($\sum F_x=0$)

$-F_{34}\sin(\phi)+G_{14} = 0$ ($\sum F_y=0$)

$\mu r_{34}F_{34}-M_{14}-\mu G_{14}a=0$ ($\sum M_B=0$)

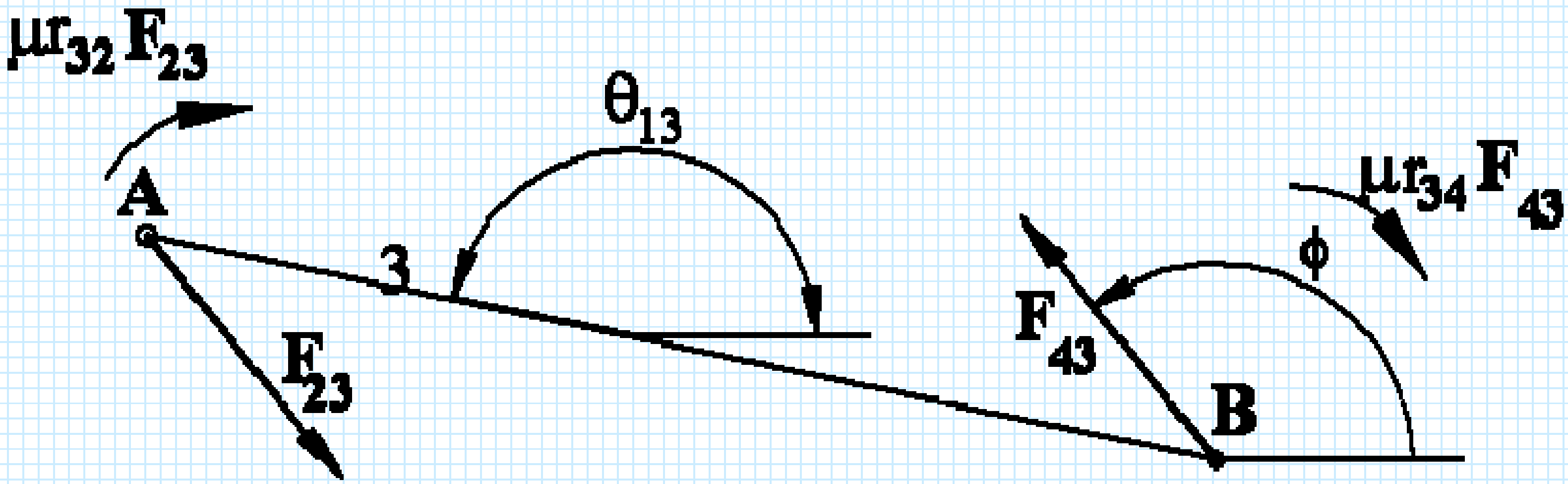


$-F_{34}\cos(\phi)-0.1G_{14}=100$ (i)

$-F_{34}\sin(\phi)+G_{14} = 0$ (ii)

$5F_{34}-M_{14}-5G_{14} =0$ (M in Nmm) (iii)

Unknowns are F_{34}, ϕ , G_{14} and M_{14}.



$\mathbf{F}_{43} = -\mathbf{F}_{34} = -\mathbf{F}_{23}$

$(\Sigma M_A = 0)$:

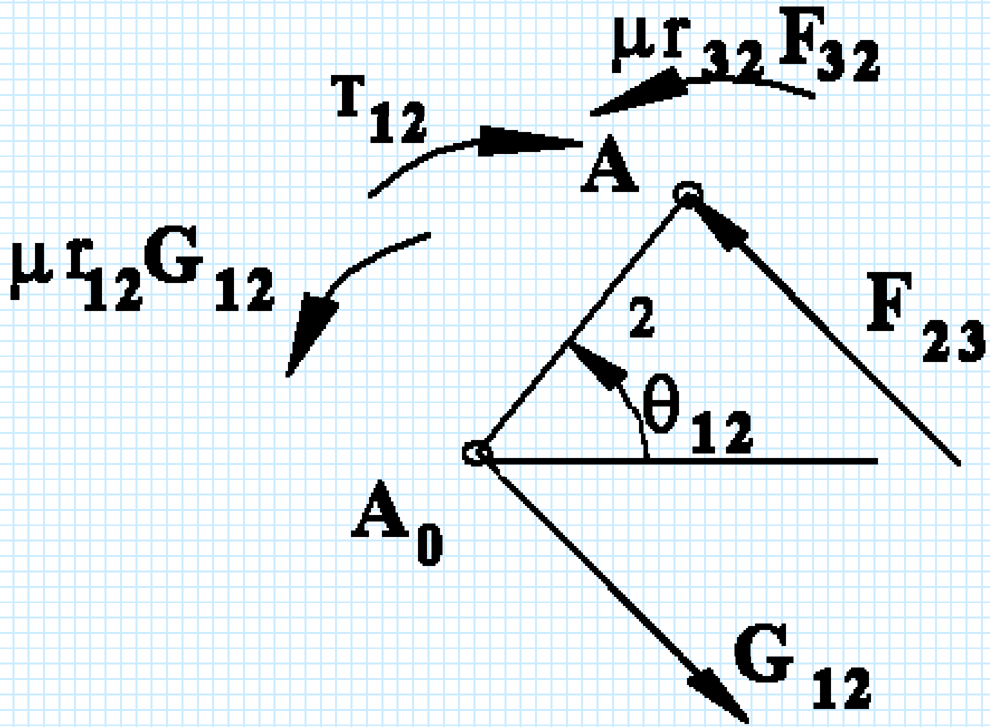
$-|AB|\, F_{43}\sin(\phi - \theta_{13}) - \mu r_{23} F_{23} - \mu r_{34} F_{43} = 0$

or:

$-800 F_{43}\sin(\phi - \theta_{13}) - 25 F_{23} - 5 F_{43} = 0$

since $F_{43} = F_{23}$ (in magnitude):

$-800 F_{43}\sin(\phi - \theta_{13}) - 30 F_{43} = 0$



$\mathbf{F}_{32} = -\mathbf{F}_{23} = -\mathbf{F}_{34} = \mathbf{F}_{43}$

$\mathbf{G}_{12} = -\mathbf{F}_{32}$

$(\sum M_{Ao}=0)$:

$|A_0A|\,F_{23}\sin(\phi-\theta_{12})+\mu r_{23}F_{23}+\mu r_{12}G_{12}+T_{12}=0$

which can be simplified as (noting $F_{43} = F_{23}$)

$200\,F_{43}\sin(\phi-\theta_{12})+30\,F_{43}+T_{12}=0$ (v)

When there are several forces acting on different links:

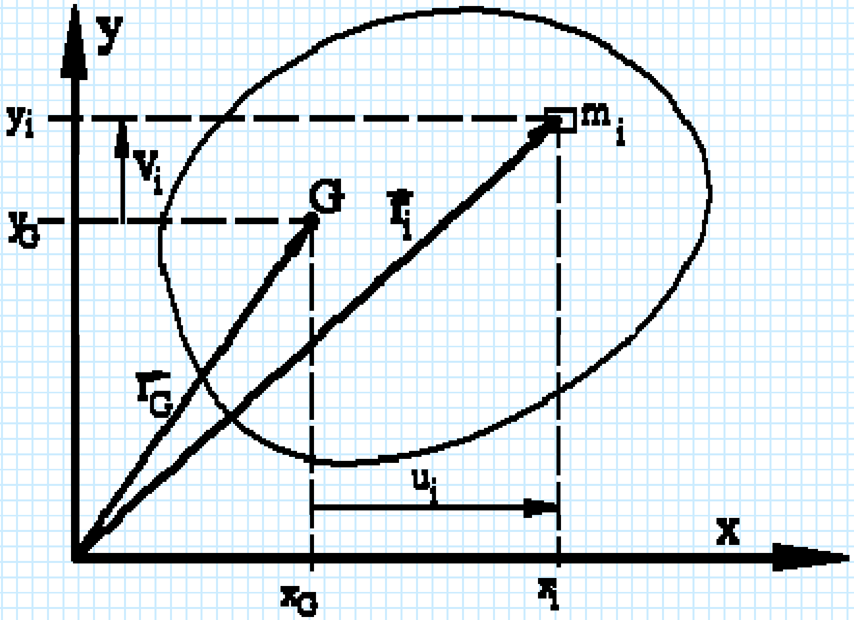
1. Principle of superposition cannot be used.
2. Some of the equations are not linear for the unknowns. Therefore, numerical iterative solutions are required.

In mechanical systems, if the designer has taken some good design measures, friction can usually be neglected in revolute joints with size small compare to link length dimensions (this usually simplifies the solution)

DYNAMIC FORCE ANALYSIS

<u>Center of Mass</u>, G, is commonly known as the ***center of gravity***,

$$\vec{r}_G = \frac{\sum_i \vec{r}_i m_i}{m} \qquad x_G = \Sigma\, x_i m_i / m \qquad y_G = \Sigma\, y_i m_i / m$$



Moment of inertia

$$I_0 = \sum_i (x_i^2 + y_i^2) m_i = \sum_i r_i^2 m_i \qquad k_0^2 = \frac{I_0}{m}$$

$$I_G = \sum_i (u_i^2 + v_i^2) m_i$$

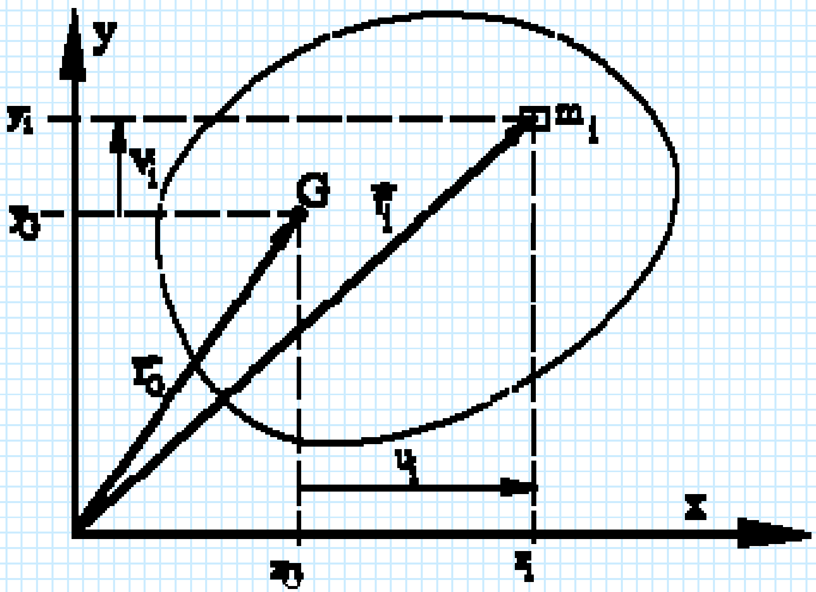
$$I_0 = \sum_i (x_i^2 + y_i^2) m_i = \sum_i \left[(x_G + u_i)^2 + (y_G + v_i)^2 \right] m_i$$

$$I_0 = \sum_i (u_i^2 + v_i^2) m_i + (x_G^2 + y_G^2) \sum_i m_i + 2 x_G \sum_i u_i m_i + 2 y_G \sum_i v_i m_i$$

$$I_0 = I_G + m r_G^2 = m(k_G^2 + r_G^2)$$

Parallel Axis Theorem

Newton's Second Law of Motion for a Rigid Body



$$\vec{F}_i + \sum_j \vec{F}_{ji} = m_i \vec{a}_i = \frac{d^2(m_i \vec{r}_i)}{dt^2}$$

$\mathbf{F}_{ji}$ is commonly known as the ***internal force***.

For all particles:

$$\sum_i \vec{F}_i + \sum_i \sum_j \vec{F}_{ji} = \sum_i m_i \vec{a}_i = \frac{d^2(\sum_i m_i \vec{r}_i)}{dt^2}$$

$$\sum_i \vec{F}_i = \sum \vec{F}$$ sum of all external forces acting.

$$\sum_i \sum_j \vec{F}_{ji} = 0 \qquad \sum_i m_i \vec{r}_i = m\vec{r}_G$$

Newton's Second Law of motion for Linear momentum

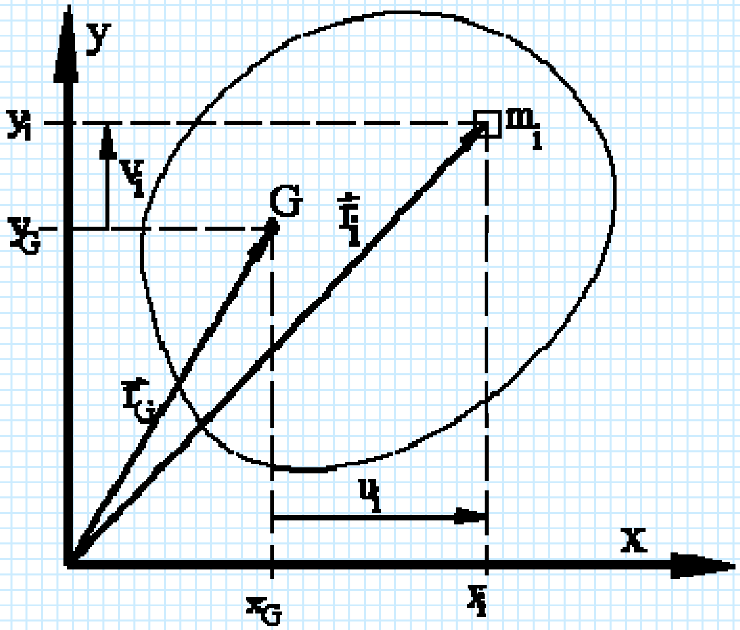
$$\sum \vec{F} = \frac{d^2(m\vec{r}_G)}{dt^2} = \frac{d(m\vec{v}_G)}{dt} = m\vec{a}_G$$

Moment equilibrium

$$\vec{r}_i x\vec{F}_i + \sum_j \vec{r}_i \vec{F}_{ji} = \vec{r}_i x\vec{a}_i m_i$$

$$\mathbf{a}_i = \mathbf{a}_0 + \mathbf{a}_{i/0} = \mathbf{a}_0 + \mathbf{a}_{i/0}{}^n + \mathbf{a}_{i/0}{}^t$$

$$\mathbf{a}_{i/0}{}^t = \alpha \text{ x } r \qquad \mathbf{a}_{i/0}{}^n = -\omega^2 \mathbf{r}_i$$



$$\sum_i \vec{r}_i x\vec{F}_i + \sum_i \sum_j \vec{r}_i \vec{F}_{ji} = \sum_i \vec{r}_i x\vec{a}_i m_i = \sum_i \left[\vec{r}_i x\left(\vec{a}_0 + \vec{\alpha} x\vec{r}_i - \omega^2 \vec{r}_i\right) m_i\right]$$

$$\sum_i \vec{r}_i x\vec{F}_i = \sum \vec{M}_0$$

$$\sum_i \sum_j \vec{r}_i x\vec{F}_{ji} = 0$$

$$\sum_i \left[\vec{r}_i x\left(\vec{a}_0 + \vec{\alpha} x\vec{r}_i - \omega^2 \vec{r}_i\right) m_i\right] = \sum_i m_i \vec{r}_i x\vec{a}_0 + \sum_i m_i \vec{r}_i x\vec{\alpha} x\vec{r}_i - \omega^2 \sum_i \vec{r}_i x\vec{r}_i m_i$$

$$\sum_i m_i \vec{r}_i x\vec{a}_0 = m\vec{r}_G x\vec{a}_0$$

$$\sum_i m_i \vec{r}_i x\vec{\alpha} x\vec{r}_i = \vec{\alpha} \sum_i r_i{}^2 m_i = I_0 \vec{\alpha}$$

$$\omega^2 \sum_i \vec{r}_i x\vec{r}_i m_i = 0$$

$$\sum \vec{M}_0 = m\vec{r}_G x\vec{a}_0 + I_0 \vec{\alpha}$$

Newton's Second Law of Motion for Angular Momentum

$$\sum M_G = \frac{d^2 I_G \vec{\theta}}{dt^2} = \frac{d(I_G \vec{\omega})}{dt} = I_G \vec{\alpha}$$

D'Alambert's Principle

$$\sum \vec{F} - m\vec{a}_G = 0$$

$$\sum M_G - I_G \vec{\alpha} = 0$$

$F^i = -m\vec{a}_G$ A nonexistant (fictituous) force ***inertia force***

$\vec{T}^i = -I_G \vec{\alpha}$ A nonexistant (fictituous) torque ***inertia torque***

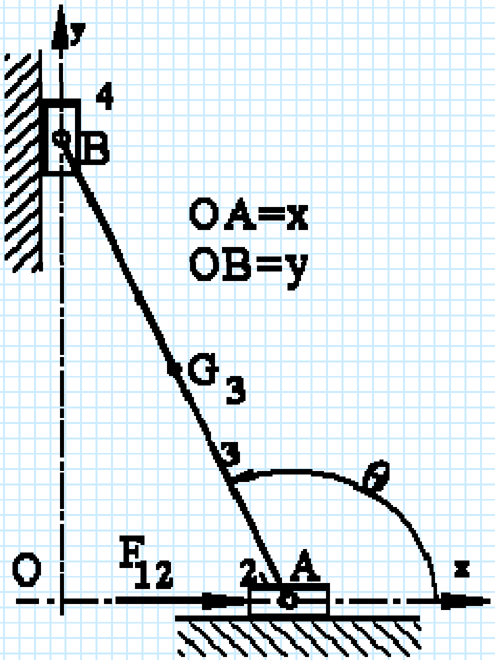
Considering Inertia force and torque as if an external force or torque

$$\sum \vec{F} = 0 \qquad \sum M_G = 0$$

D'Alambert's Principle

In a body moving with a known angular acceleration and a linear acceleration of the center of gravity, the vector sum of all the external forces and inertia forces and the vector sum of all the external moments and inertia torque are both separately equal to zero.

Example



$$x + a_3 e^{i\theta} = iy$$

$$\cos\theta = -\frac{x}{a_3} \qquad \omega_{13} = \frac{\dot{x}}{a_3 \sin\theta}$$

$$y = a_3 \sin\theta \qquad \dot{y} = a_3 \omega_{13} \cos\theta$$

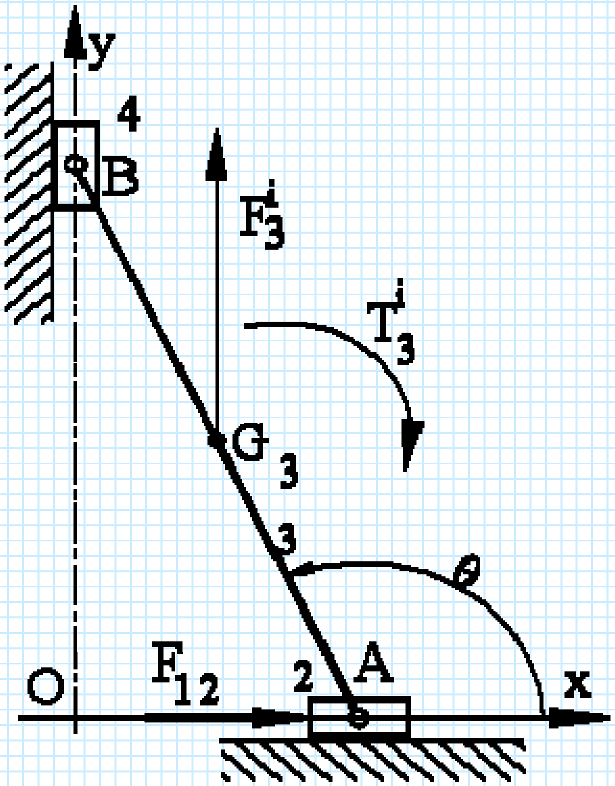
$$\alpha_{13} = \frac{-\omega_{13}\dot{x}\cos\theta}{a_3 \sin^2\theta}$$

$$\ddot{y} = -a_3\omega_{13}^2 \sin\theta + a_3\alpha_{13}\cos\theta$$

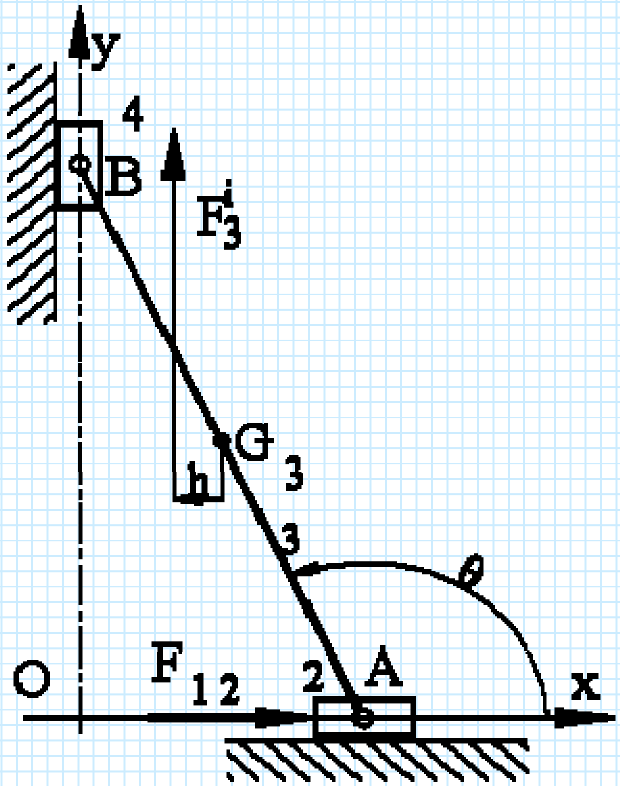
$$\vec{r}_{G3} = x + \frac{1}{2} a_3 e^{i\theta} \qquad F_3^i = -m\vec{a}_{G3}$$

$$\vec{v}_{G3} = \dot{x} + \frac{1}{2} i a_3 \omega_{13} e^{i\theta} \qquad T_{G3}^i = -I_{G3}\alpha_{13} =$$

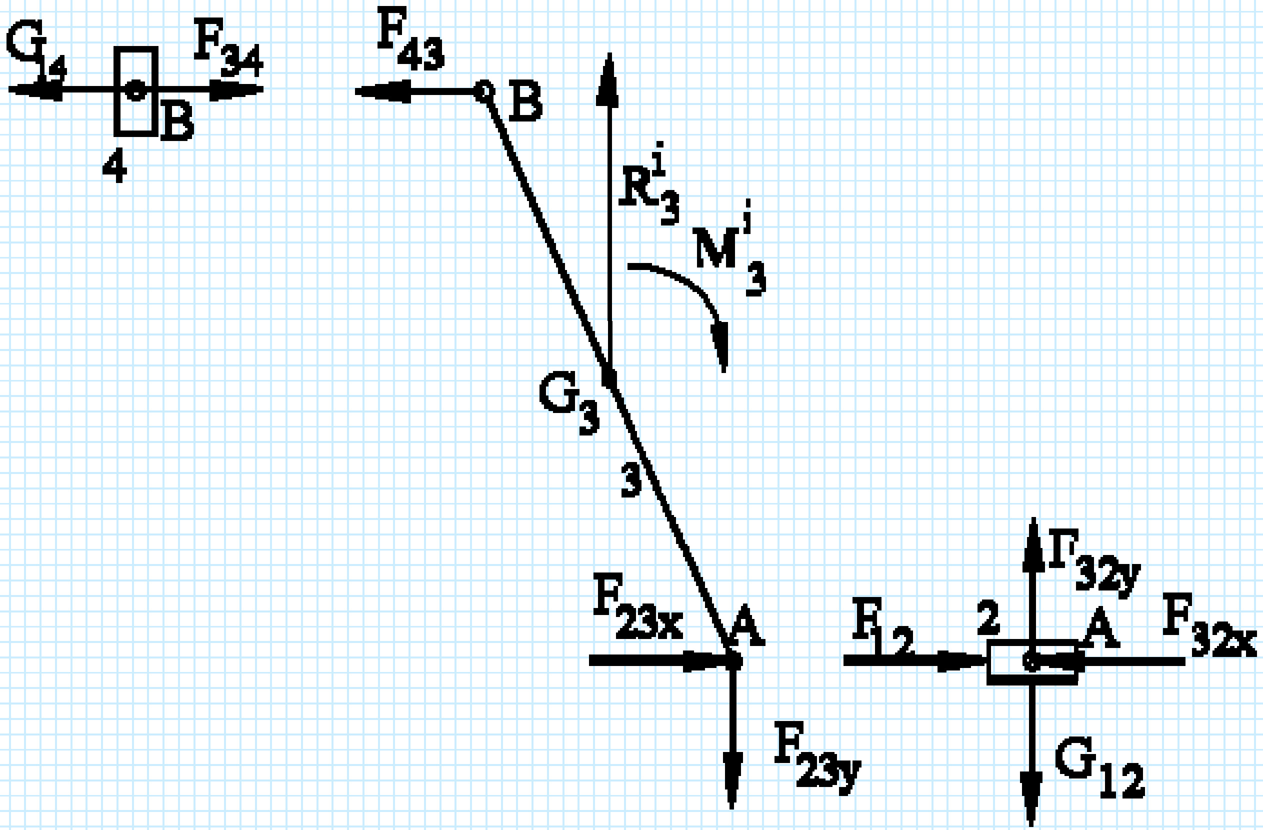
$$\vec{a}_{G3} = \ddot{x} + \frac{1}{2} a_3 e^{i\theta}(i\alpha_{13} - \omega_{13}^2)$$



(a)



(b)

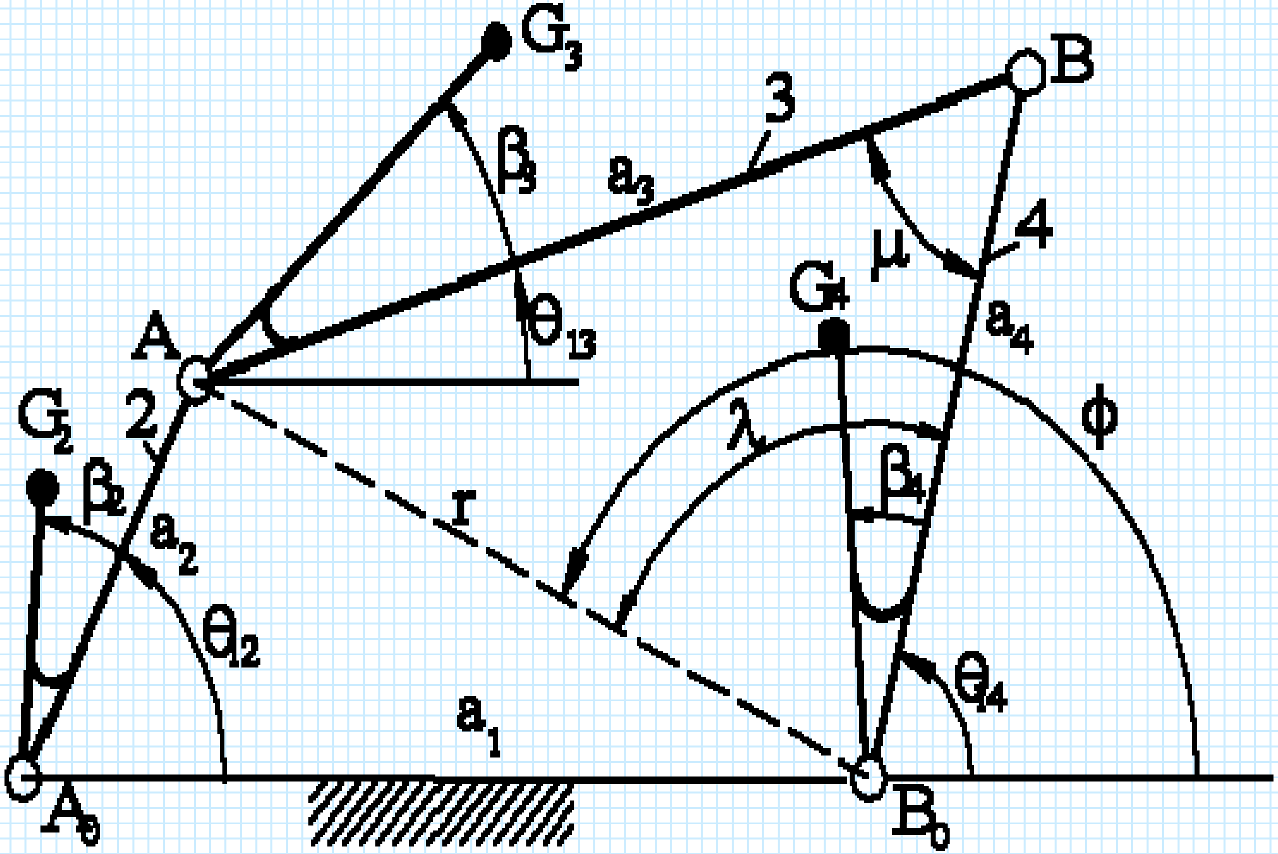


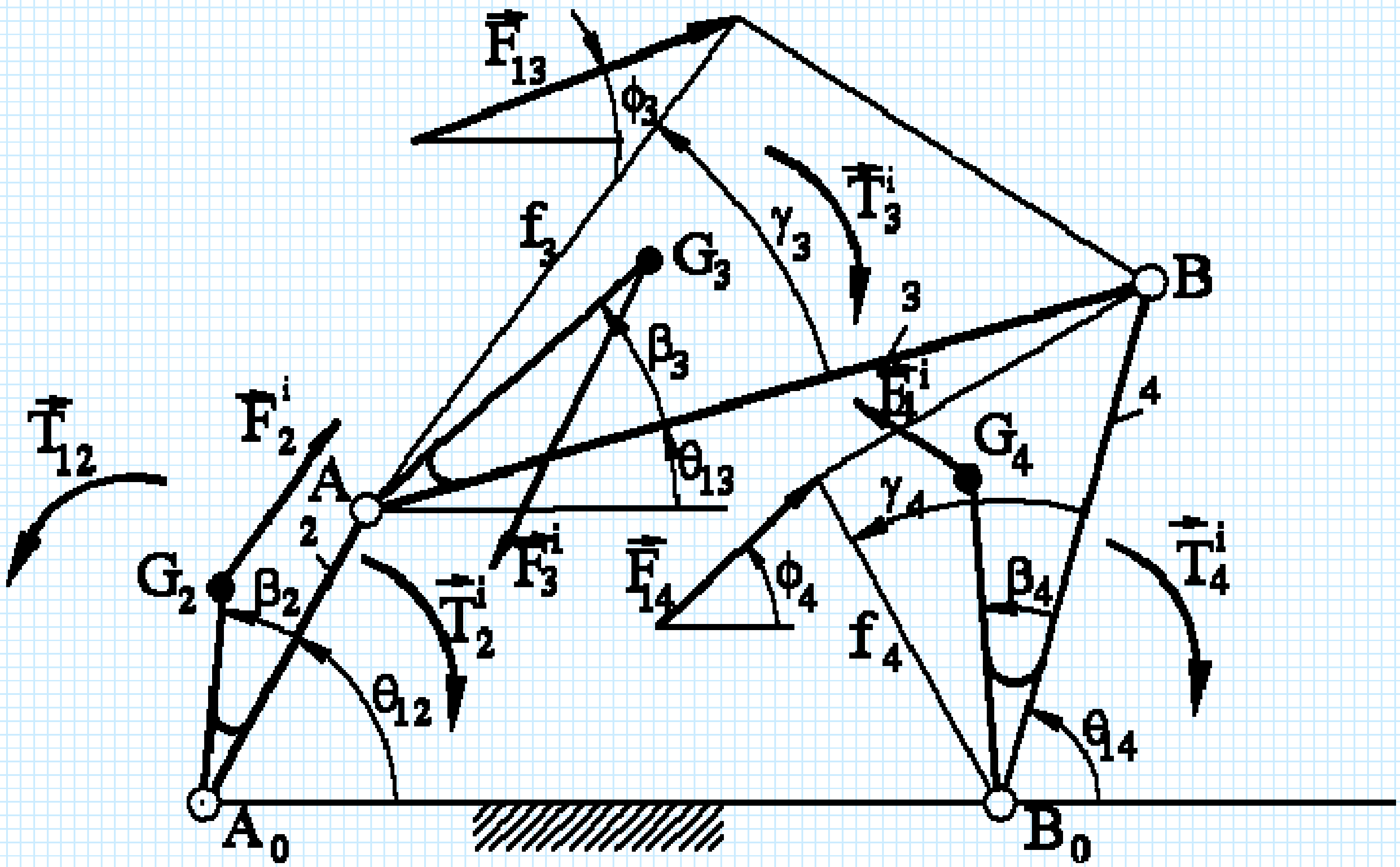
For Link 3

$F_{43} - F_{23x} = 0$

$R_3^i - F_{23y} = 0$

$500F_{43}\sin(180-113.578) + 250(25.97)\sin(90-113.578)-865 = 0$


G3
B
3
β3
a3
4
μ
G4
a4
A
θ13
ϕ
G2
2
λ
β2
r
β4
a2
θ2
θ14
a1
A0
B0



$\vec{F}_{13}$
ϕ_3
$\vec{T}_3^i$
γ_3
f_3
G_3
B
3
β_3
$\vec{F}_4^i$
4
$\vec{T}_{12}$
F_2^i
A
θ_{13}
G_4
2
$\vec{F}_3^i$
$\vec{F}_{14}$
γ_4
$\vec{T}_4^i$
G_2
β_2
$\vec{T}_2^i$
ϕ_4
β_4
f_4
θ_{12}
θ_{14}
A_0
B_0