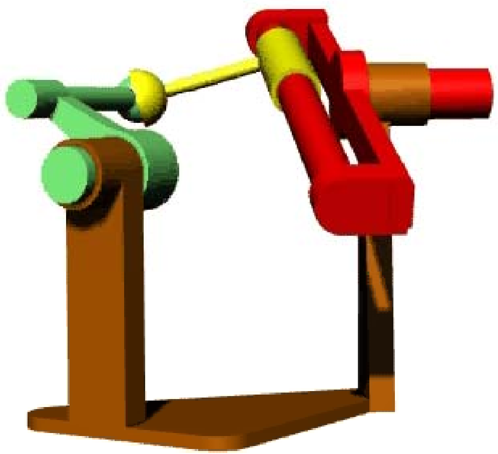
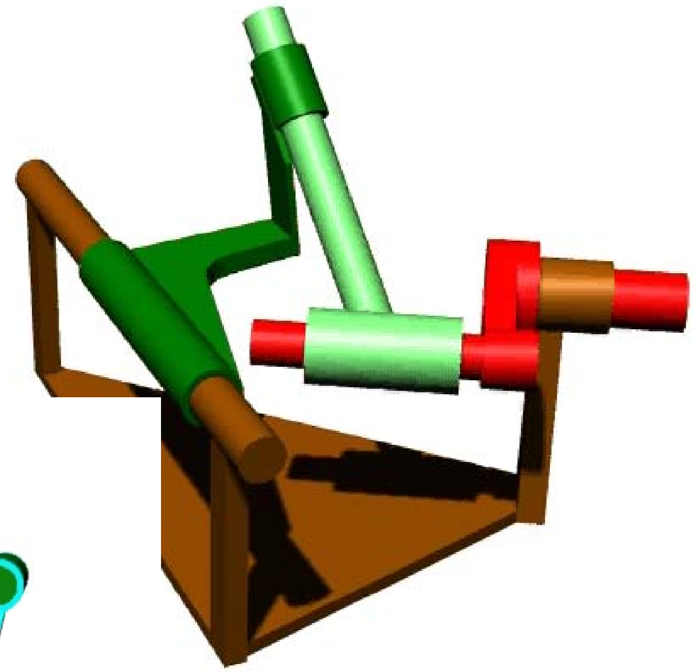
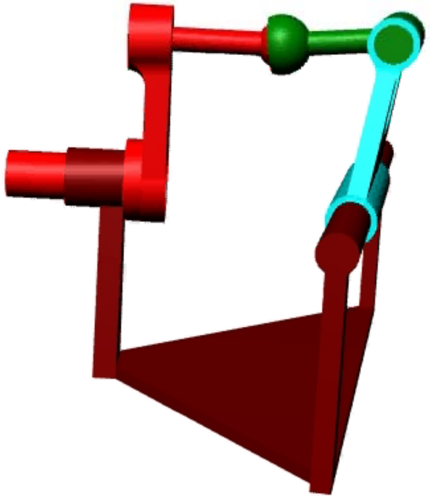






ME301 THEORY OF MACHINES 1- MECHANISMS

KINEMATIC ANALYSIS OF MECHANISMS

We shall consider planar mechanisms only.

In this chapter we shall assume that we know the dimensions of all the links. If the mechanism has **F degrees of freedom**, we shall assume that we know the value of **F number of parameters.**

Our aim is:

1. Determine the position of all the links in the mechanism
2. Determine the paths of points on these links
3. To determine velocity and acceleration characteristics of all the links or points on these links.

Position: Location of a rigid body (link) or a particle (point) in a rigid body with respect to a given reference frame.

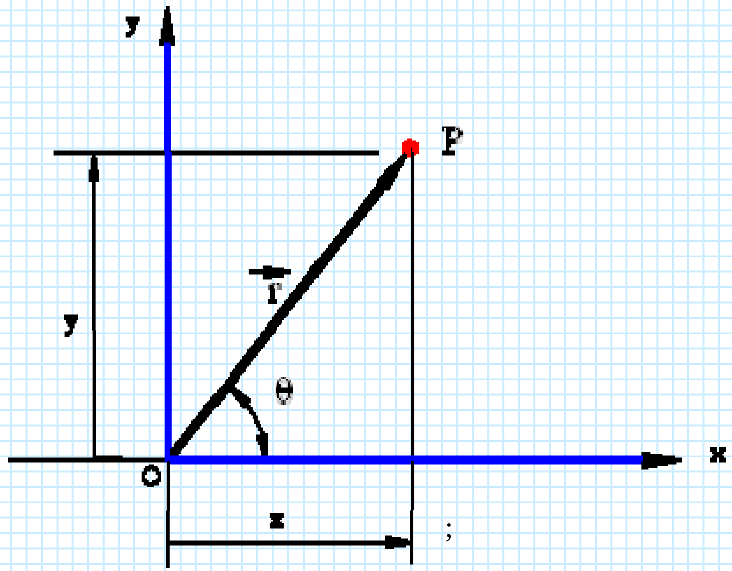
Path: Locus of successive positions of a particle (point) on a rigid body (link).

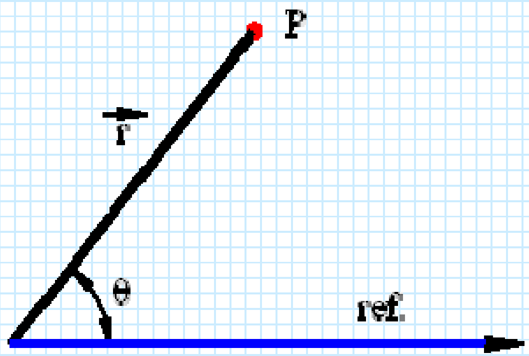
Displacement: Change in position of a rigid body (link) or a particle (point) with respect to a reference frame. It is a vector quantity whose magnitude is called ***distance*** (measured in mm or m).

Velocity: The rate of change of position of a particle or a rigid body. It is the time rate of change of displacement. It is a vector quantity whose magnitude is called speed (mm/sec = mms^{-1} or m/sec = ms^{-1}).

Acceleration: Time rate of change of velocity. It is a vector quantity whose magnitude is measured in $mm/sec^2 = mms^{-2}$ or $m/sec^2 = ms^{-2}$.

Kinematics of a Particle





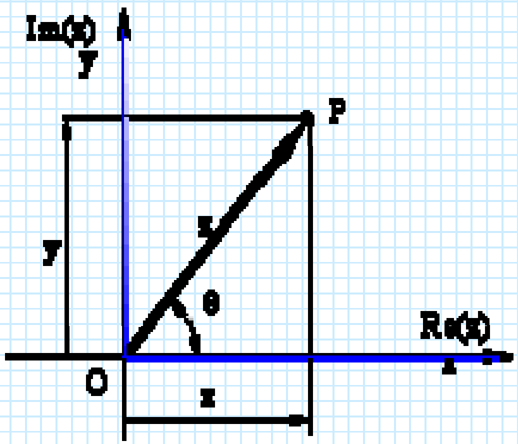
x=rcosθ   y=rsinθ

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \tan^{-1}(\frac{y}{x})$$

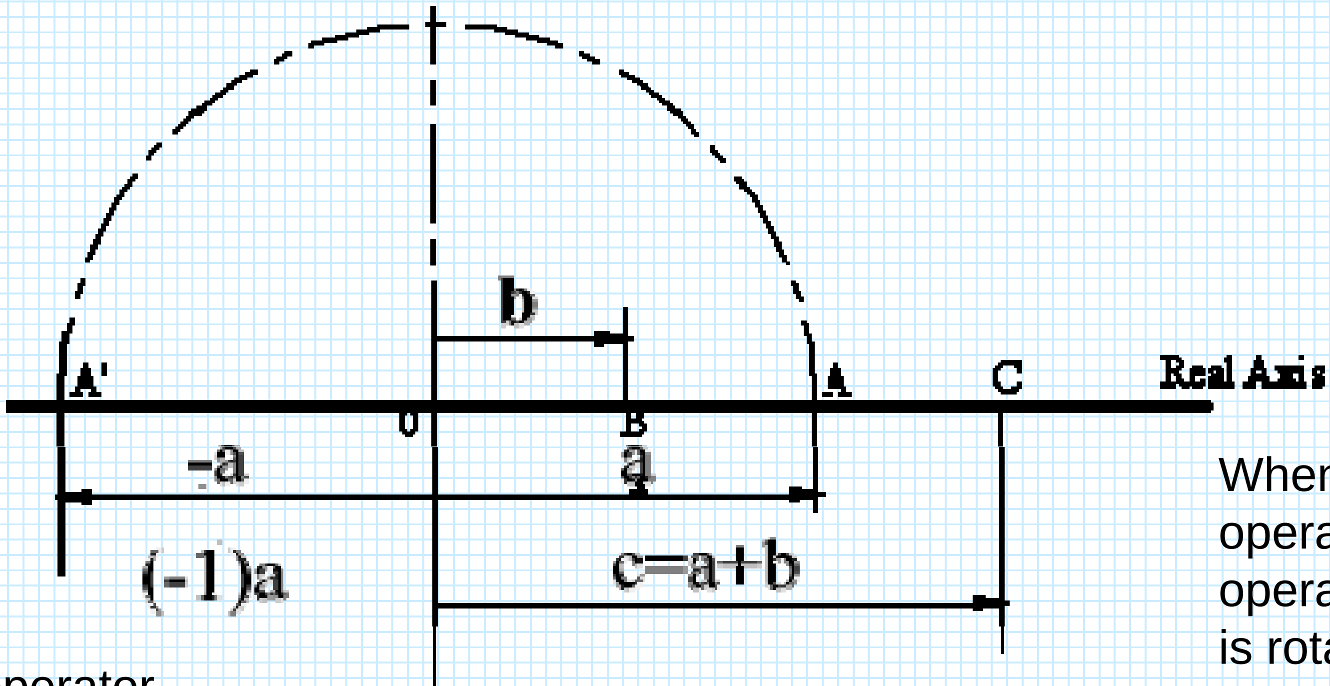
$$\vec{r} = \hat{i}x + \hat{j}y$$

$$\vec{r} = r\angle\theta$$



Complex Numbers

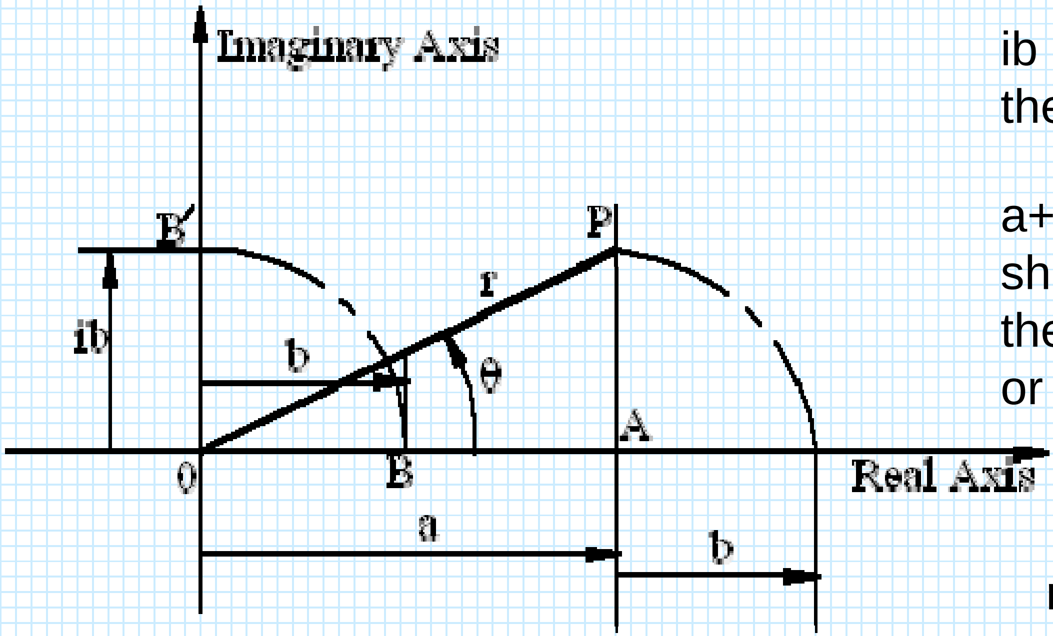
Real Numbers: are used to represent the magnitude of a quantity



When a real number is operated by (-1) operator, that number is rotated by 180^0

(-1) operator.

(i) operator: when this operator operates on a real number, that number is to be rotated by 90^0 CCW.



ib is the “imaginary number”, shows the real number b rotated 90^0 CCW.

a+ib is the “complex number”. It shows the location of a point P in the complex plane (Cauchy plane or Gauss-Argand Diagram)

r is the *modulus*, θ is the argument of the complex number.

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}(b/a)$$

İf we operate on a real number by i twice: (i*i)b, the real number must rotate twice by 90^0=180^0. Since 1800 rotation is defined by (-1) operator:

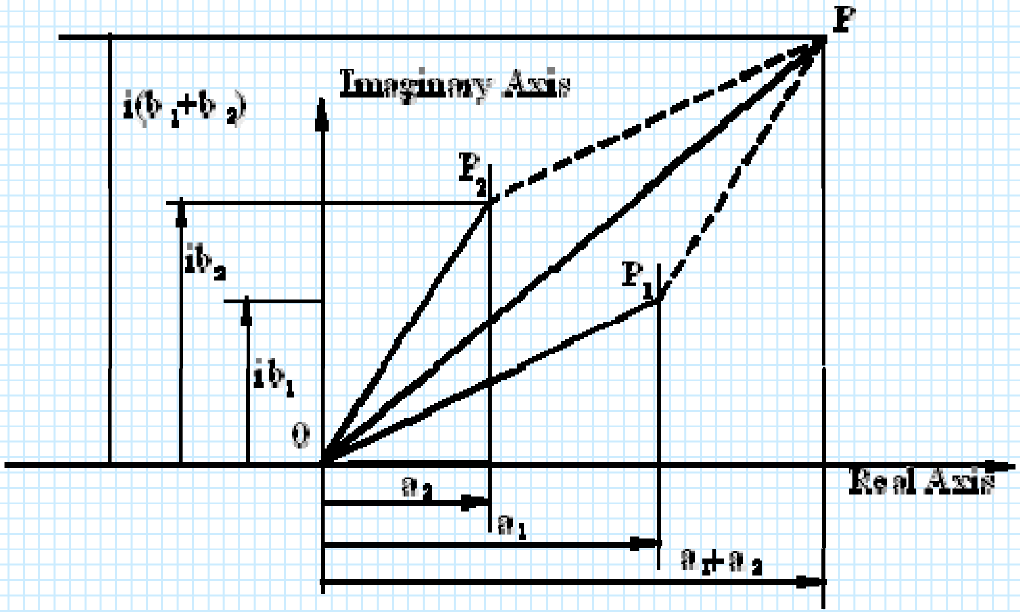
$i*i = i^2 = -1$

1. Two Complex numbers can be equal only if their real and imaginary parts are equal.

2. Complex numbers add vectorially (Paralelogram Law of addition)

The sum of two complex numbers is determined by adding real and imaginary parts separately. If $\mathbf{c_1}= a_1+ib_1$ and $\mathbf{c_2}= a_2+ib_2$ then

$$\mathbf{z=c_1+c_2}= (a_1+a_2)+i(b_1+b_2)$$



3. Multiplication and division follows the rules of ordinary algebra with the additional relation $i^2=-1$.

c=*a*+*ib* : *ortogonal representation*

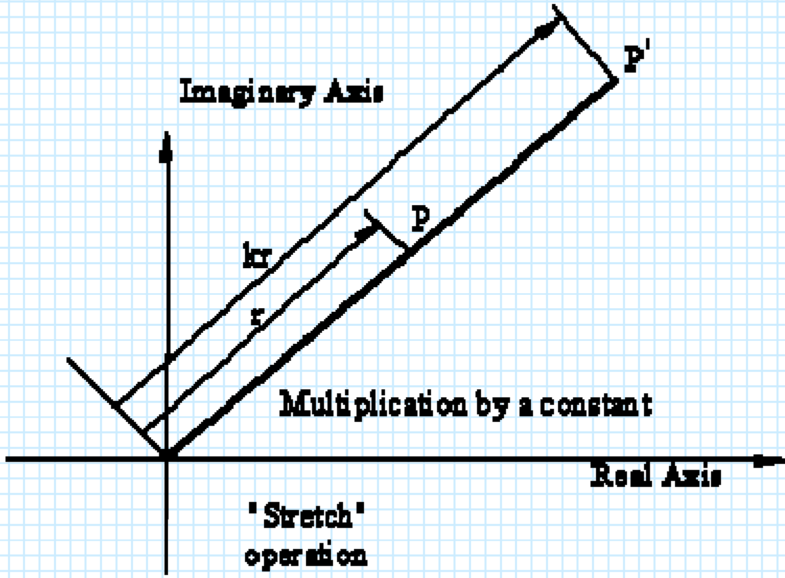
c=r(cosθ+isinθ)

Euler's equation: $e^{i\theta} = \cos\theta + i \sin\theta$,

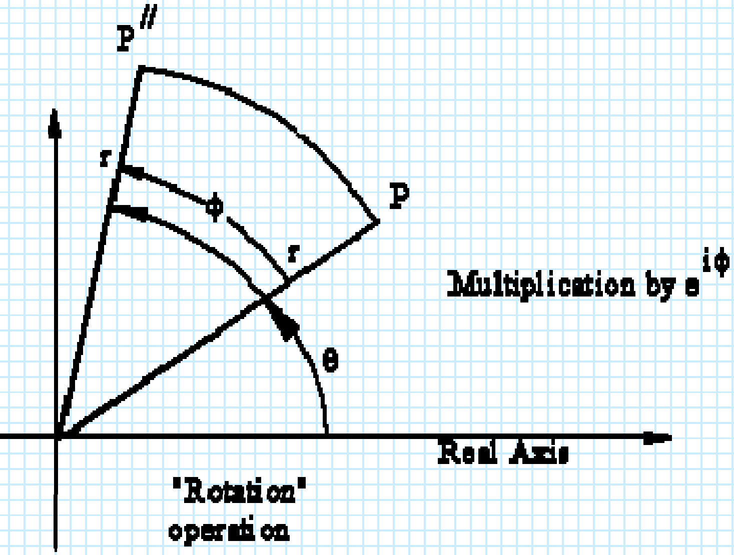
c =r $e^{i\theta}$: Exponential form

u= $e^{i\theta}$ a unit vector making an angle θ wr to real axis

Multiplication of a complex number by a constant, k



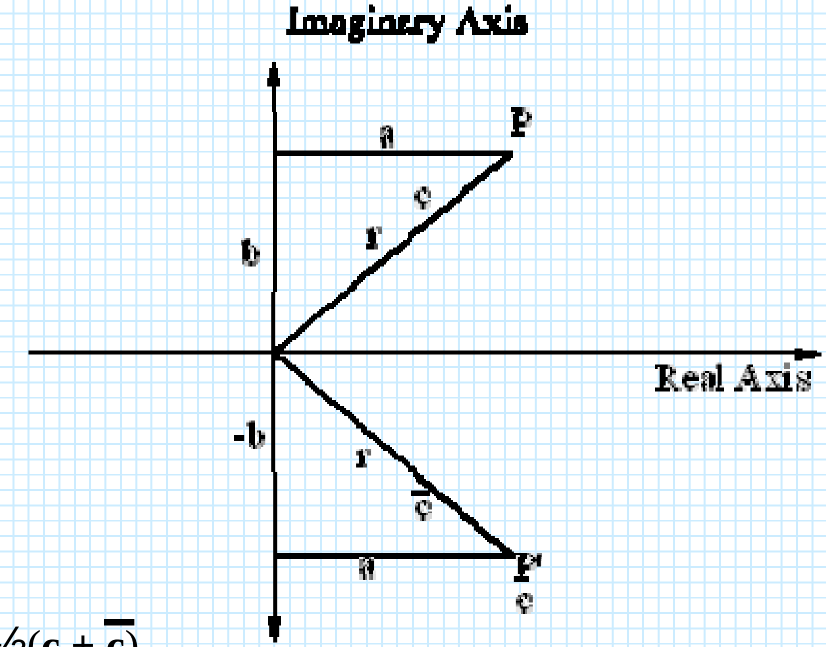
Multiplication of a complex number by $e^{i\phi}$.



Conjugate of a complex number:

$\mathbf{c}=a+ib$ $\mathbf{c}= re^{i\theta}$

$\bar{\mathbf{c}}=a - ib$ $\bar{\mathbf{c}}= re^{-i\theta}$



$r^2 = \mathbf{c}\bar{\mathbf{c}} = (a+ib)\,(a-ib) = a^2 + b^2$

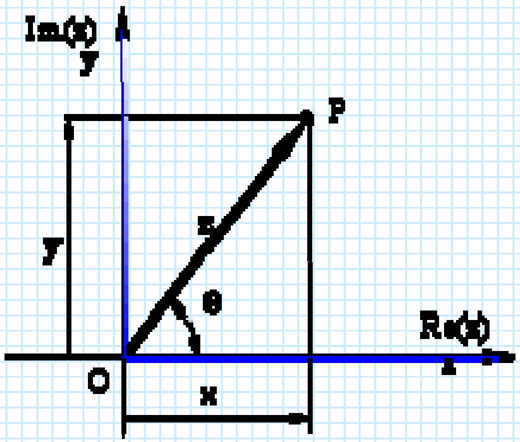
Real part of $\mathbf{c}$ =Re [$\mathbf{c}$] = ½[(a+ib) +(a-ib)] = ½($\mathbf{c} + \bar{\mathbf{c}}$)

Imaginary part of $\mathbf{c}$ = Im [$\mathbf{c}$] = ½[(a+ib)- (a-ib) = ½[(($\mathbf{c} - \bar{\mathbf{c}}$)

Differentiation of complex numbers also follows the rules of ordinary calculus.

$$z = r\cos\theta + ir\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

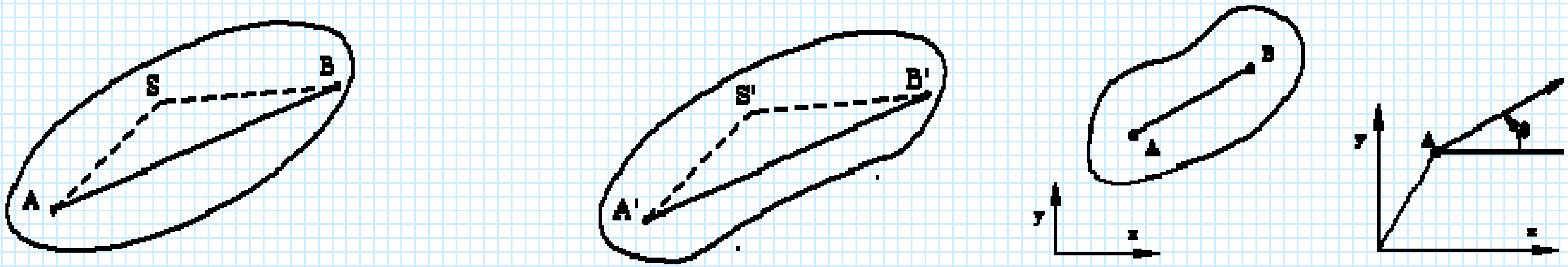


$$z = re^{i\theta}$$

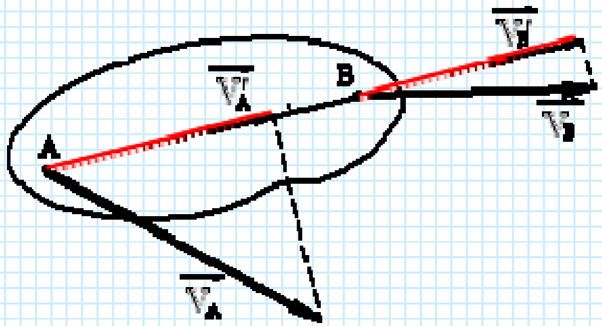
Kinematics of Rigid Body in Plane

The assumption of rigidity results with the following three important conclusions:

1. The plane motion of a rigid body is completely described by the motion of any two points within the rigid body or by a point and the angle a line on the rigid plane makes wr to a reference

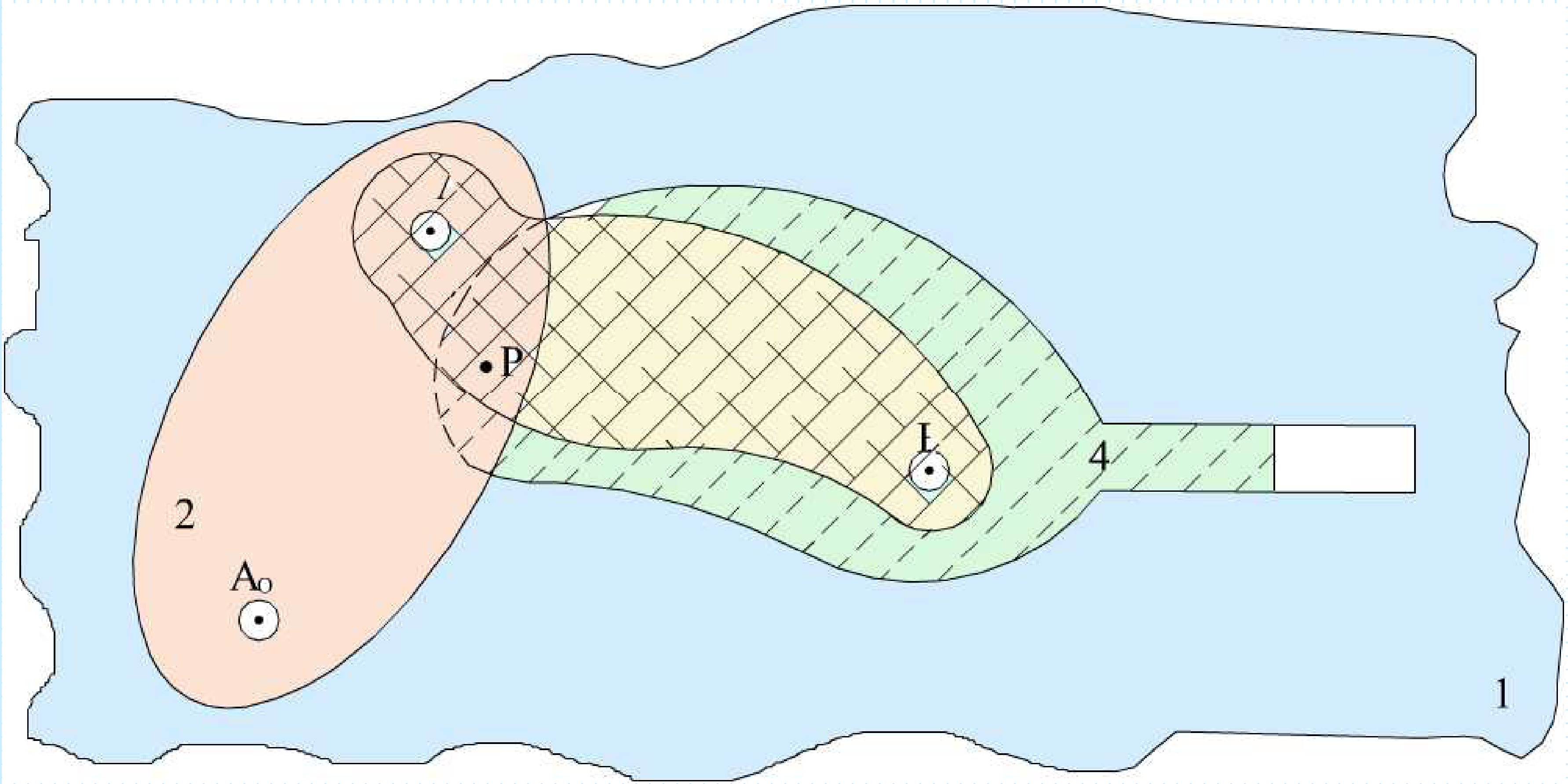


2. Rigidity ensures that the particles lying on a straight line have equal velocity components in the direction of this line, since the distance between any two points along this line remains constant.



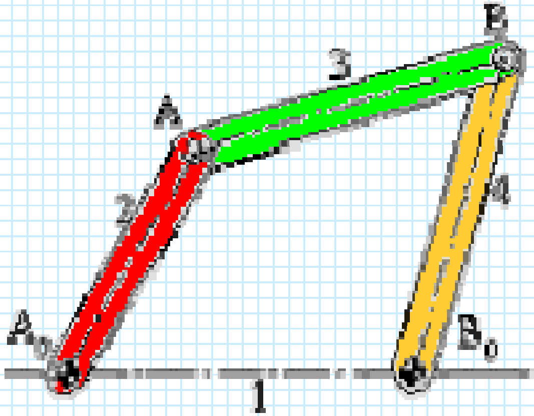
3. We are concerned with the kinematics of the rigid bodies only. It is sufficient to consider just a line on the rigid body (vector **AB**, for example). Since the actual boundaries of the body does not influence the kinematics, the rigid body in plane motion is to be regarded as a large plane which embraces any desired point in the plane.

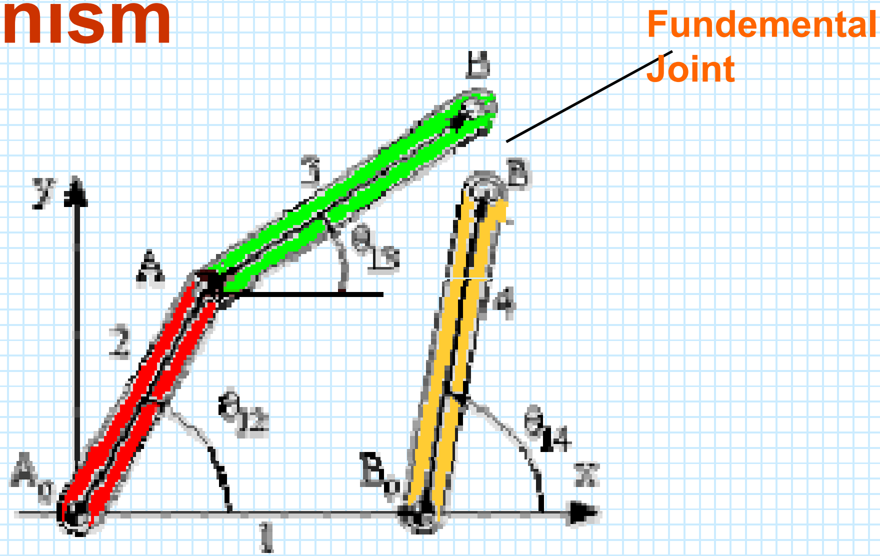
Coincident Points



02

Vector Loops of a Mechanism





$A_0B_0=a_1$, $AA_0=a_2$, $AB=a_3$, $B_0B=a_4$

$\theta_{12}, \theta_{13}, \theta_{14}$ are the position variables.

$A_0A + AB = A_0B$ (for open kinematic chain 1,2,3)

$A_0B_0 + B_0B = A_0B$ (for open kinematic chain 1,4)

$A_0A + AB = A_0B_0 + B_0B$ ***loop closure equation (vector loop equation)***

$$a_2 \cos\theta_{12} \vec{i} + a_2 \sin\theta_{12} \vec{j} + a_3 \cos\theta_{13} \vec{i} + a_3 \sin\theta_{13} \vec{j} = a_1 \vec{i} + a_4 \cos\theta_{14} \vec{i} + a_4 \sin\theta_{14} \vec{j}$$

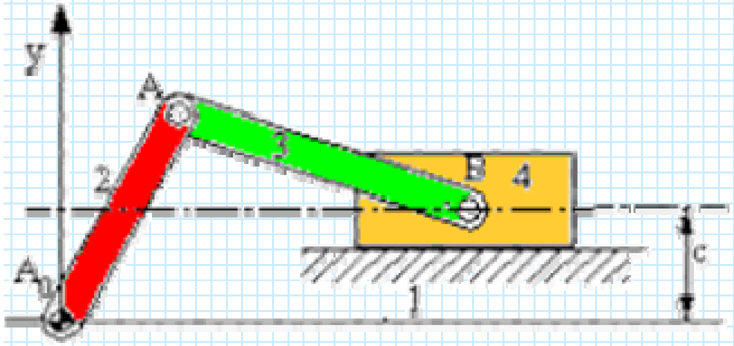
Loop closure equation in cartesian form

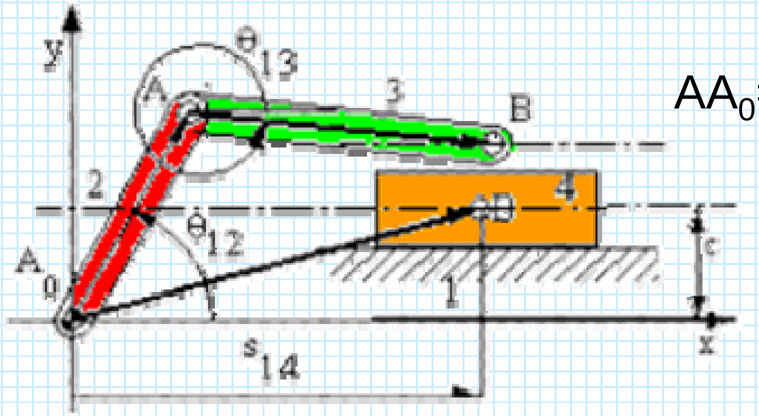
$$a_2 \cos\theta_{12} + a_3 \cos\theta_{13} = a_1 + a_4 \cos\theta_{14}$$

$$a_2 \sin\theta_{12} + a_3 \sin\theta_{13} = a_4 \sin\theta_{14}$$

$$a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = a_1 + a_4 e^{i\theta_{14}}$$

Loop Closure equation in complex numbers





$AA_0=a_2$, $AB=a_3$

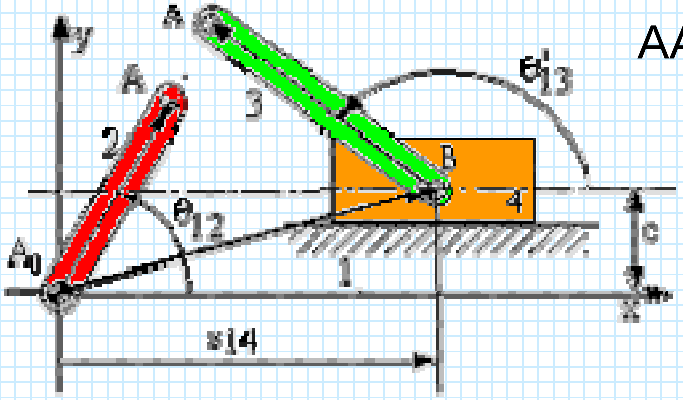
θ_{12},θ_{13}, s_{14} are the position variables.

A_0A +AB=A_0B (for open kinematic chain 1,2,3)

A_0B (for open kinematic chain 1,4)

$$\mathbf{A_0A} + \mathbf{AB} = \mathbf{A_0B}$$ ***loop closure equation (vector loop equation)***

$$a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = s_{14} + ic$$ ***Loop Closure equation in complex numbers***



$AA_0 = a_2$, $AB = a_3$

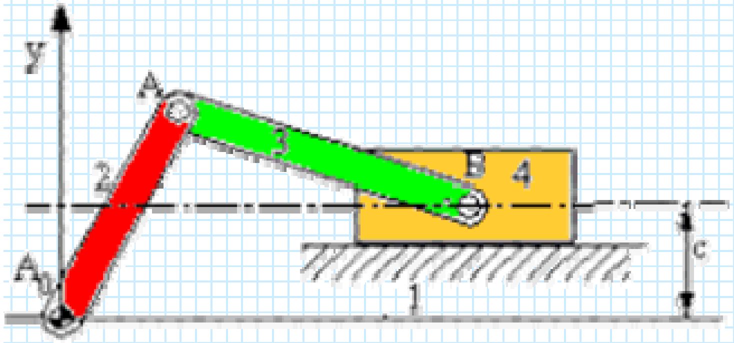
$$\theta'_{13} = \theta_{13} - \pi$$

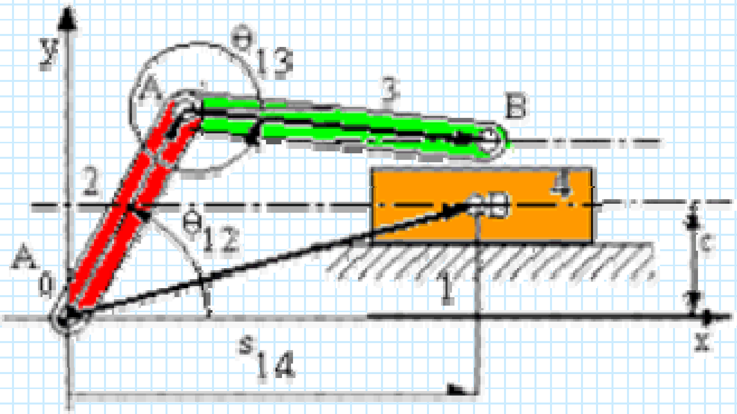
$$A_0A = A_0B + BA$$

loop closure equation (vector loop equation)

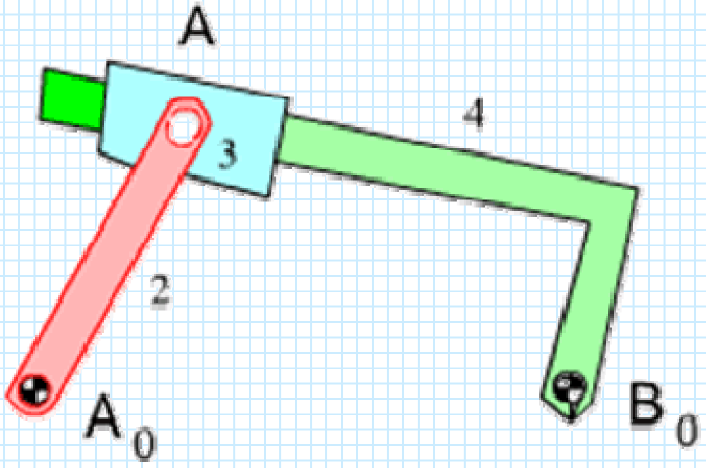
$$a_2 e^{i\theta_{12}} = s_{14} + ic + a_3 e^{i\theta'_{13}}$$

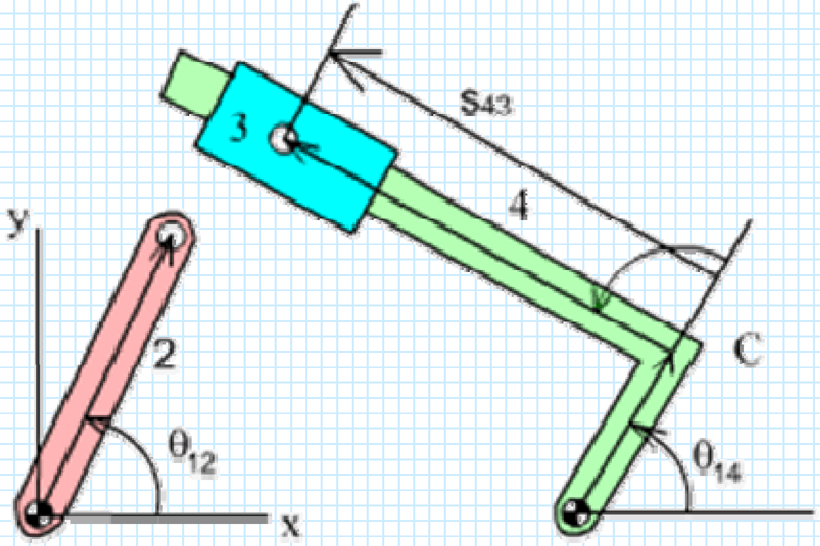
Loop Closure equation in complex numbers





Inverted Slider Crank Mechanism





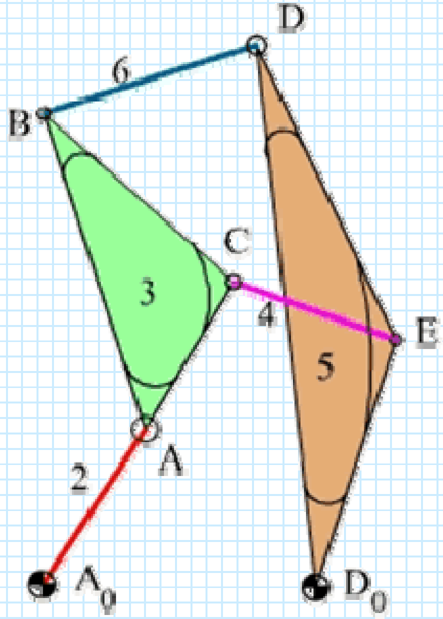
$AA_0=a_2,\ B_0C=a_4,\ A_0B_0=a_1$

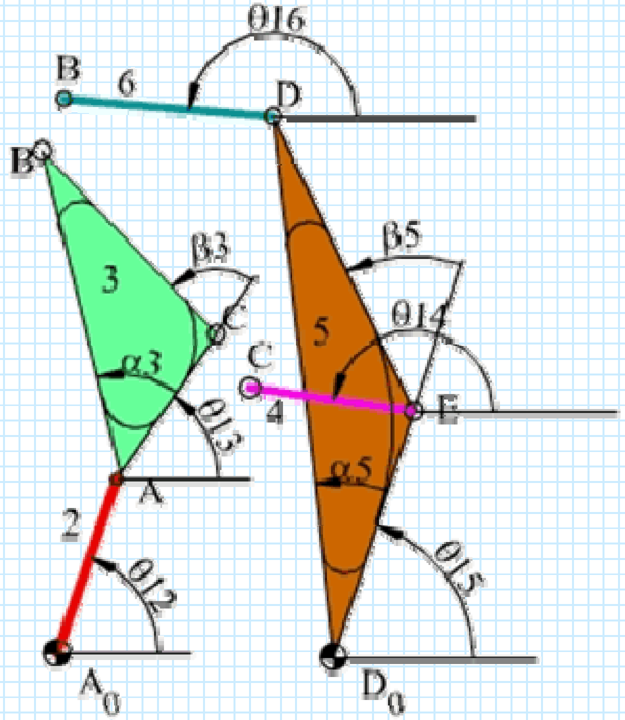
$$\mathbf{A_oA = A_oB_o + B_oC + CA}$$

$$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14}} + s_{43}e^{i(\theta_{14}+\alpha_4)}$$

05

Two loops!





$A_0D_0=a_1$, $A_0A=a_2$, $AC=a_3$, $AB=b_3$, $CB=c_3$, $EC=a_4$, $D_0E=a_5$, $D_0D=b_5$, $ED=c_5$, $BD=a_6$

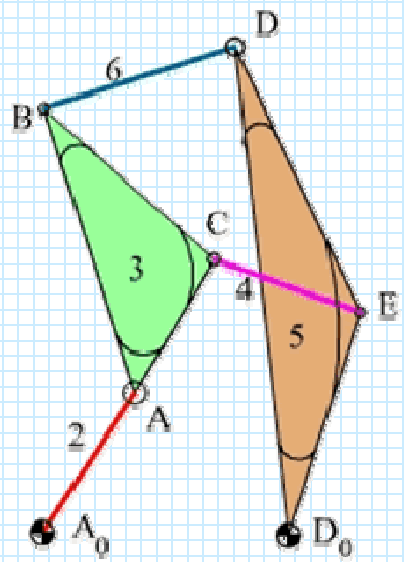
$A_0A + AC = A_0D_0 + D_0E + EC$

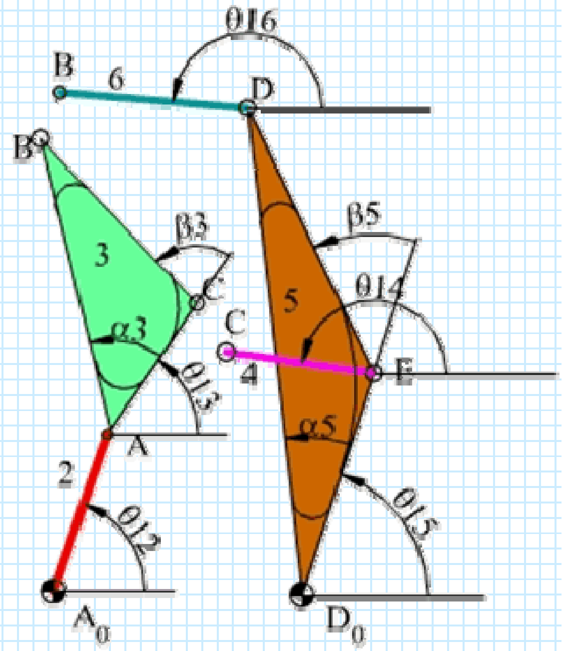
$$a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = a_1 + a_5 e^{i\theta_{15}} + a_4 e^{i\theta_{14}}$$

$A_0A + AB = A_0D_0 + D_0D + DB$

$$a_2 e^{i\theta_{12}} + b_3 e^{i(\theta_{13}+\alpha_3)} = a_1 + b_5 e^{i(\theta_{15}+\alpha_5)} + a_6 e^{i\theta_{16}}$$

A third Loop equation





EC +CB=ED+DB

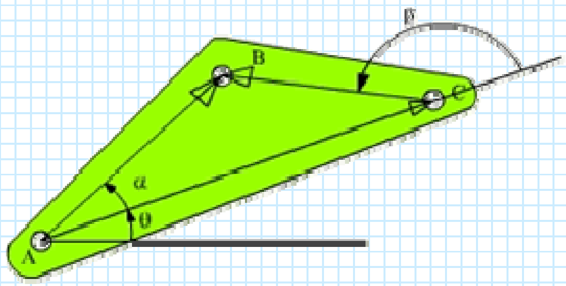
$$a_4 e^{i\theta_{14}} + c_3 e^{i(\theta_{13}+\beta_3)} = c_5 e^{i(\theta_{15}+\beta_5)} + a_6 e^{i\theta_{16}}$$

A_0A +AC=A_0D_0+D_0E +EC

Add

A_oA + AC + EC + CB =D_oE+ EC + ED +DB

EC +CB=ED+DB



AC+CB = AB

Similarly

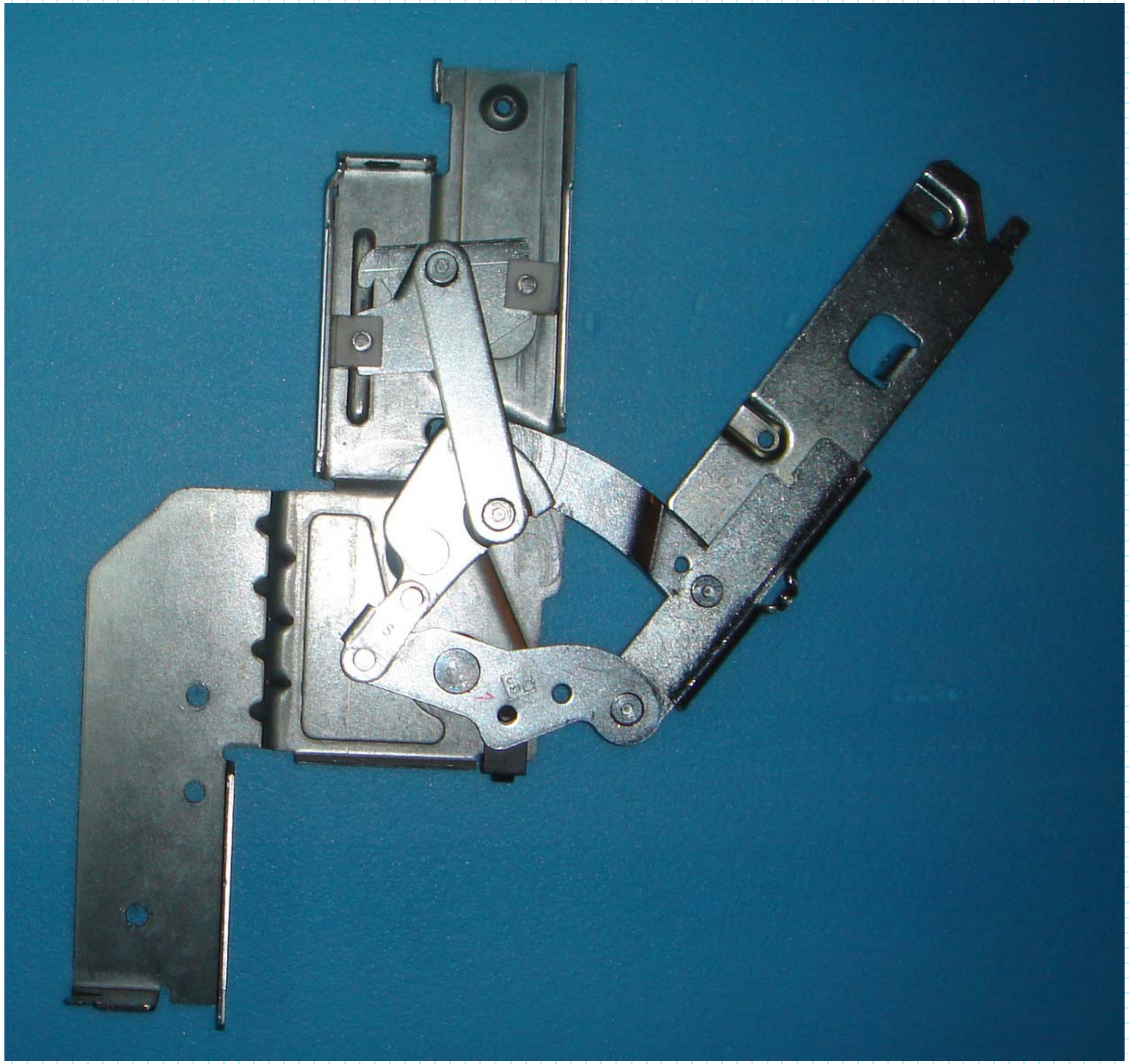
D_0E+ED = D_0D

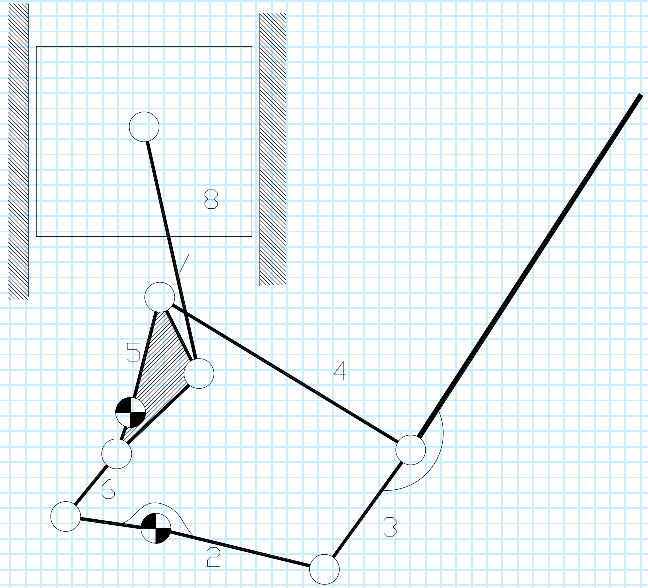
These are not loop closure Eqns.

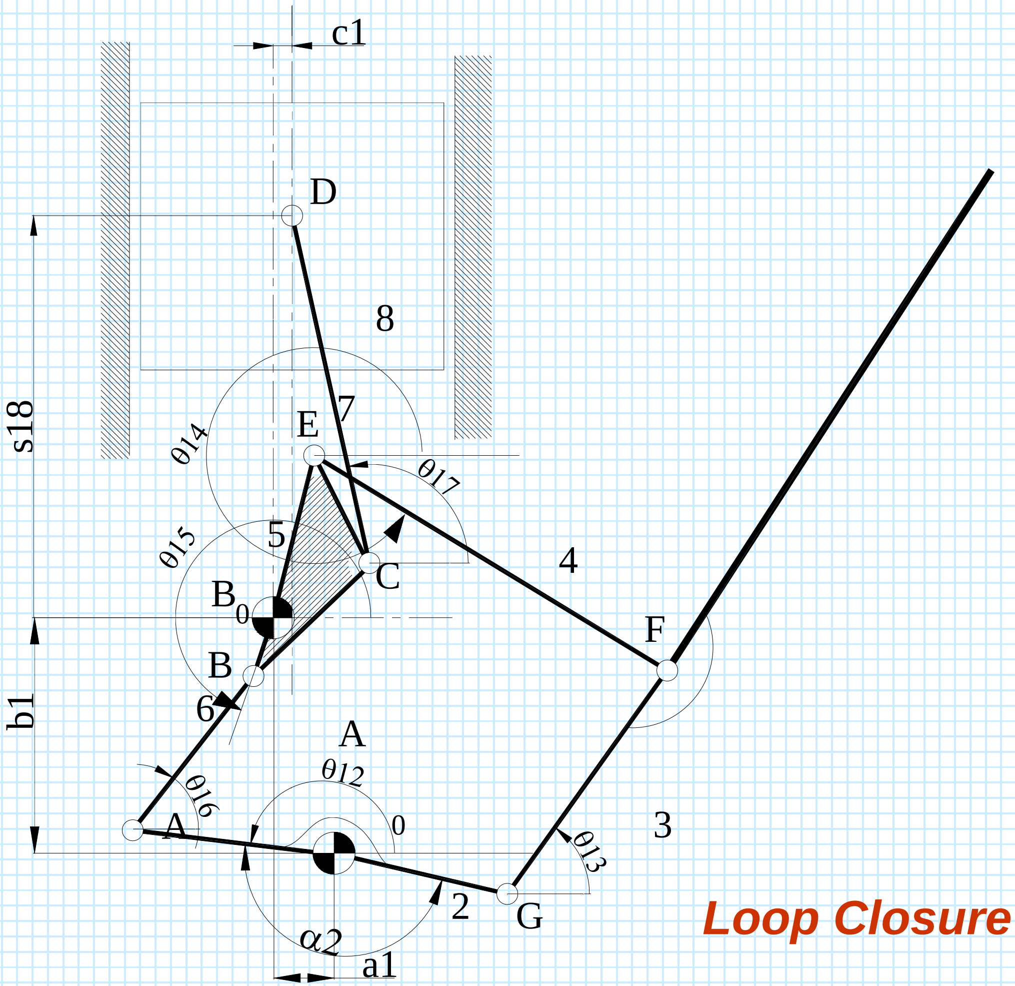
There are only 2 independent loops

A_0A +AB=A_0D_0+ D_0D +DB

Door Mechanism for a Dishwasher







$A_0A=a_{2,}$; $A_0G=b_2$; $GF=a_3$; $BA=a_6$;
$B_0B=a_5$ $B_0C=b_5$; $B_0E=c_5$; $EF=a_5$;
$CD=a_5$; $<BB_0C=\alpha_5$; $<BB_0E=\beta_5$

Vector LoopEquations:

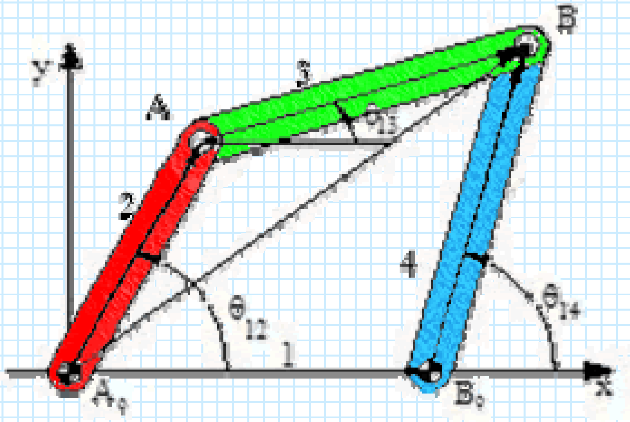
$A_0A+AB=A_0B_0+B_0B$
$A_0G+GF=A_0B_0+B_0E+EF$
$B_0C+CD=B_0D$

Loop Closure equations in complex numbers

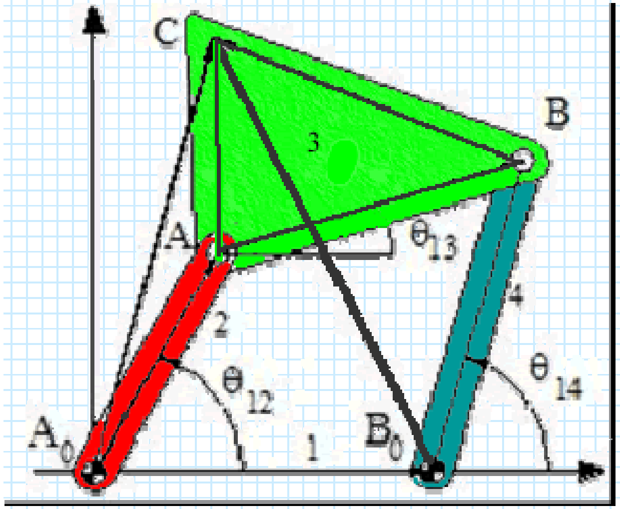
$$a_2e^{i\theta_{12}}+a_6e^{i\theta_{16}}=-a_1+ib_1+a_5e^{i\theta_{15}}$$

$$b_2e^{i(\theta_{12}+\alpha_2)}+a_3e^{i\theta_{13}}=-a_1+ib_1+c_5e^{i(\theta_{15}+\beta_5)}+a_4e^{i\theta_{14}}$$

$$b_5e^{i(\theta_{15}+\alpha_5)}+a_7e^{i\theta_{17}}=c_1+is_{18}$$



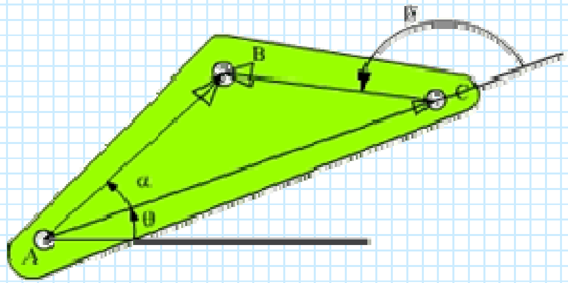
$A_0A + AB = A_0B$



$A_oA + AC = A_oC$

These are

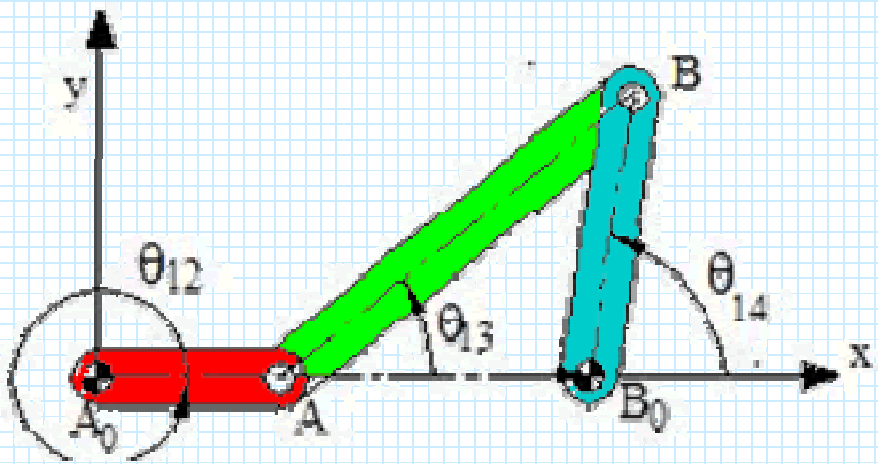
not loop equations



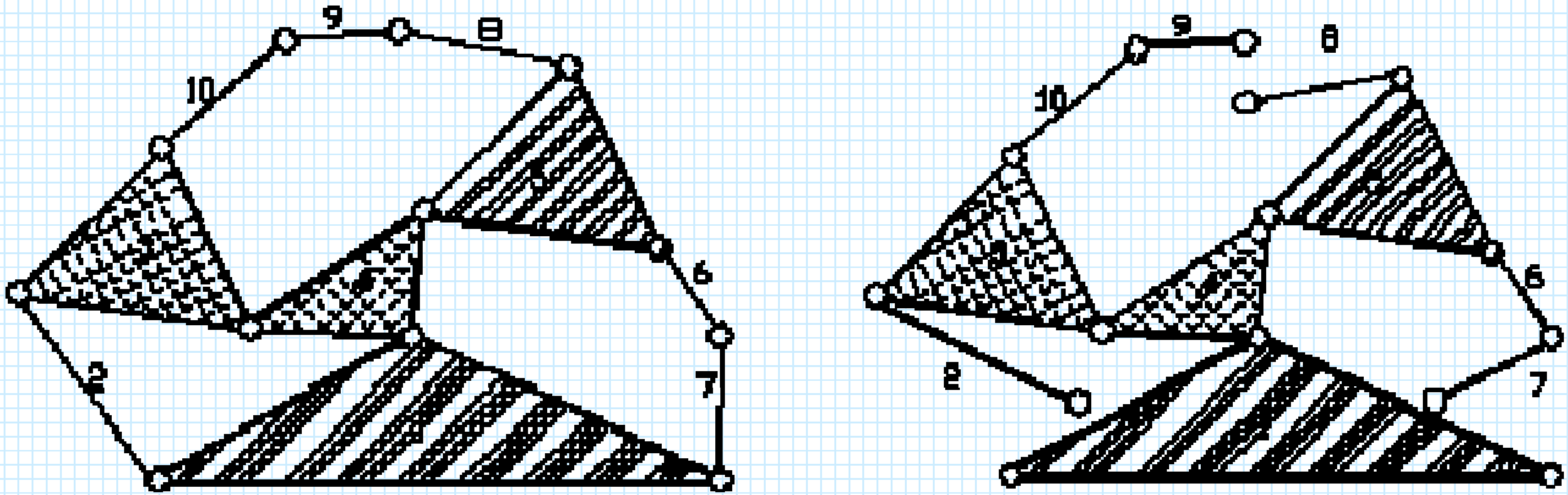
$AC+CB = AB$

Hints

1. Only **one variable angle** must be used to define the angular orientation of a link.
2. Use a_j, b_j, c_j for the fixed link lengths and α_j, β_j, γ_j for the fixed angles θ_{1j} for the variable link angles and s_{jk} for the variable lengths.
3. Beware of special positions at which the mechanism is drawn.



Euler's Equation of Polyhedra

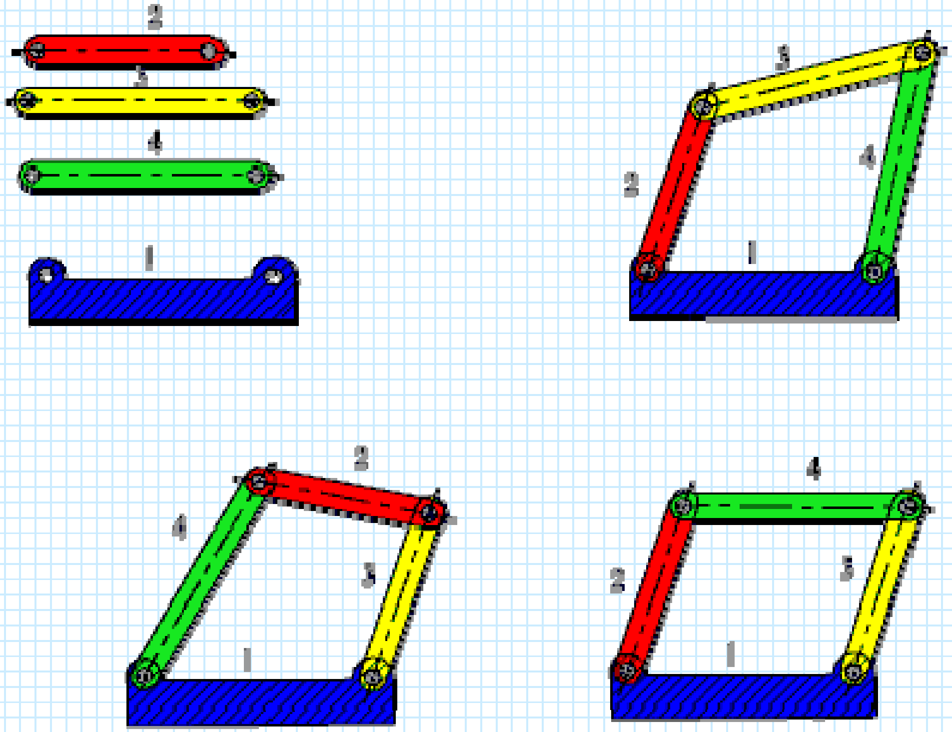


j = the number of joints in the open kinematic chain + the number of joints removed.

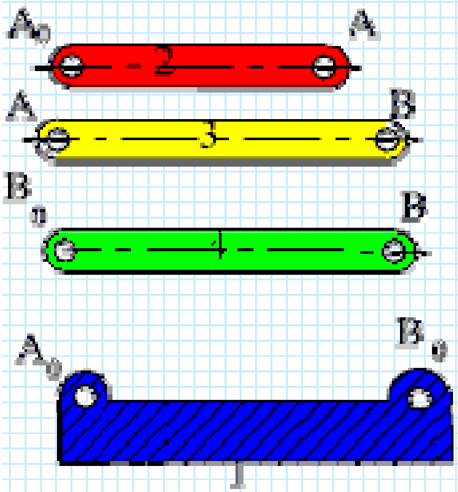
$j = (l\text{-}1) + L$ or

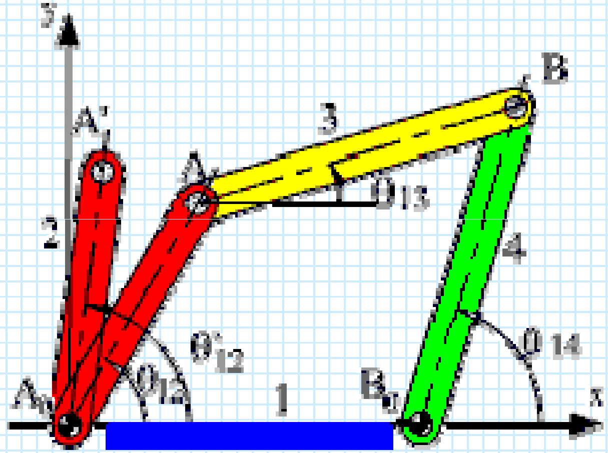
$L = j - l + 1$ **(Euler's Equation of polyhedra)**

Graphical Solution:



If you are given 4 links, you can combine them in 8 different ways

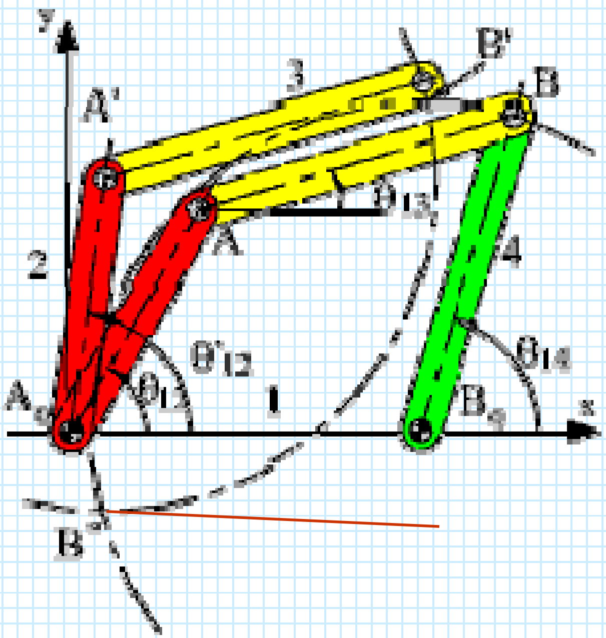




Given one form of assembly determine the position of the links when the independent parameter changes its value from θ_{12} to θ_{12}'

$$A_0A+AB = A_0B_0+B_0B$$

$$A_0A'+AB' = A_0B_0+B_0B'$$



$A_0B_0 + A_0A' = B_0A'$

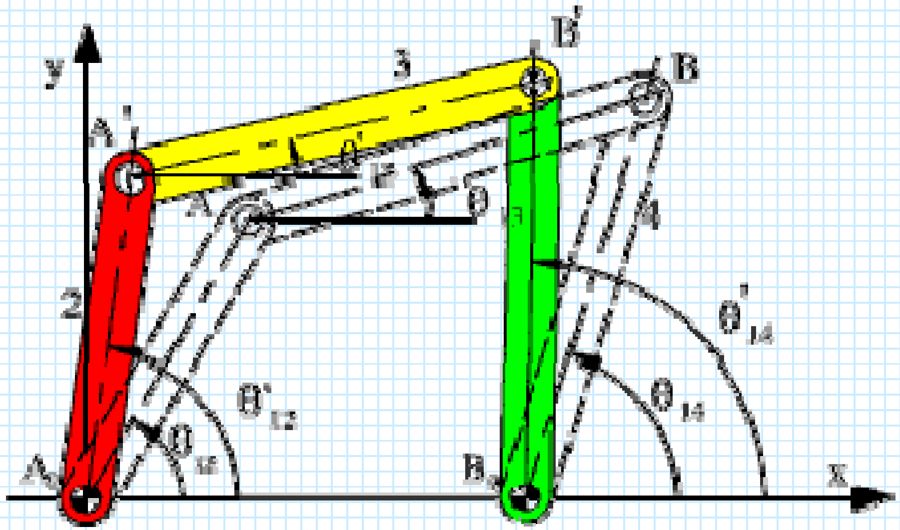
Both vectors are known

Solve the vector equation

$B_0A' + A'B' = B_0B'$

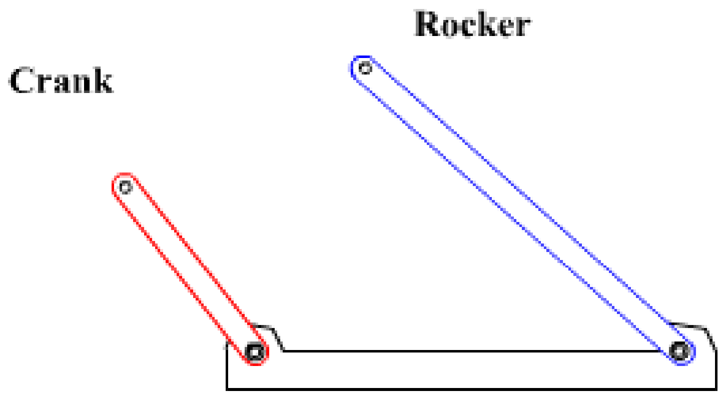
The magnitudes of the three vectors are known

What about B"???



Grashof's Rule

1. The link may have a full rotation about the fixed axis ***(crank)***
2. *The link may oscillate (swing) between two limiting angles (**rocker**).*



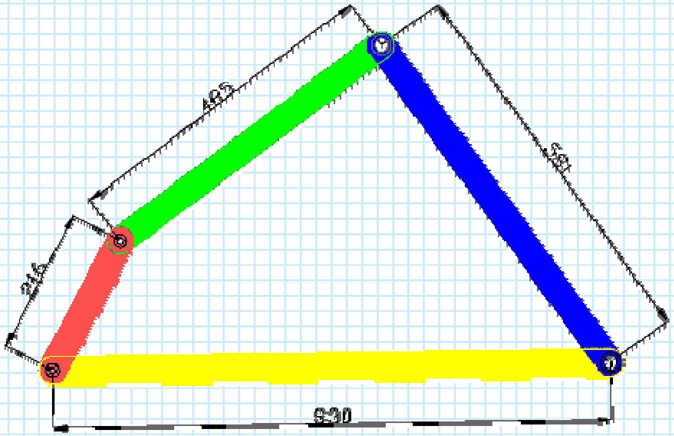
3 possibilities for a four-bar mechanism:

i) Both of the links connected to the fixed link can have a full rotation. This type of four-bar is called ***"double-crank*** *"* or "***drag-link***. "

j) Both of the links connected to the fixed link can only oscillate. This type of four-bar is called ***"double-rocker***."

k) One of the links connected to the fixed link oscillates while the other has a full rotation. This type of four-bar is called "***crank-rocker***".

l= length of the longest link
s= length of the shortest link
p,q = length of the two intermediate links



l=830, s=216 , p=485 , q=581

830 + 216 = 1046 < 485+581 = 1066

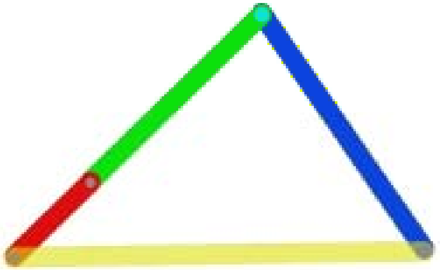
IF *l* + s < p + q

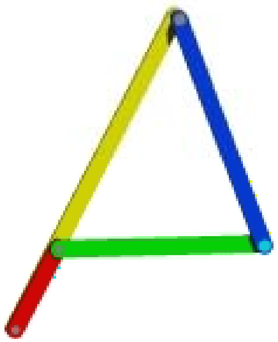
a),b) Two different crank-rocker mechanisms are possible. In each case the shortest link is the crank, the fixed link is either of the adjacent links.

c) One double-crank (drag-link) is possible when the shortest link is the frame.

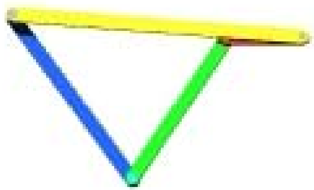
d) One double-rocker mechanism is possible when the link opposite the shortest link is the frame.

$$l + s < p + q$$

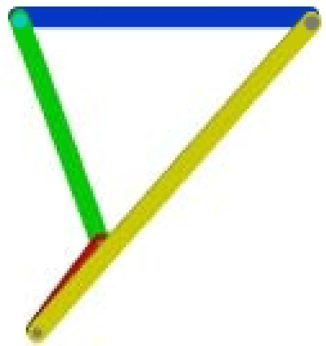




Crank-Rocker



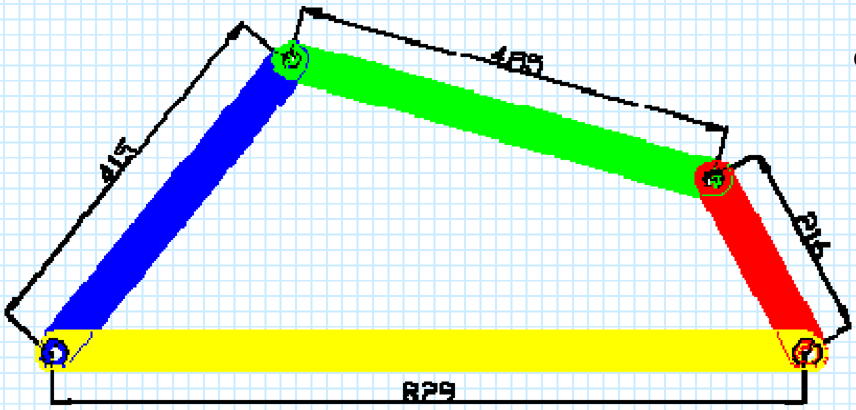
Drag-Link



Double Rocker

$l + s > p + q$

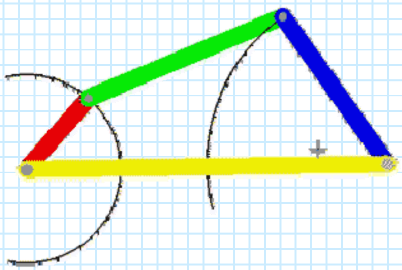
Only double-rocker mechanisms are possible (four different mechanisms, depending on the fixed link).

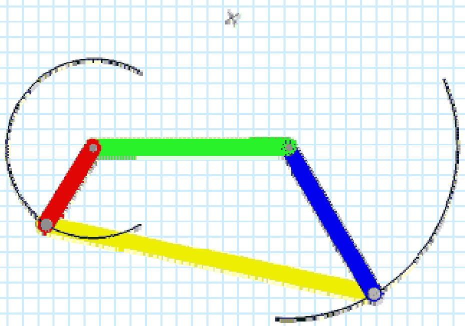


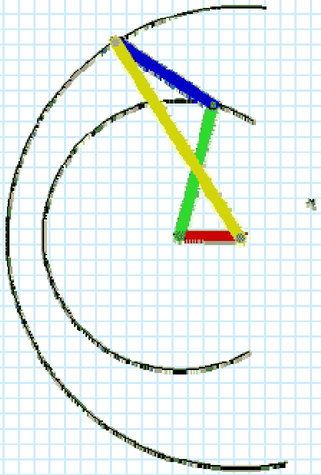
l=829, s=216 , p=485 , q=415

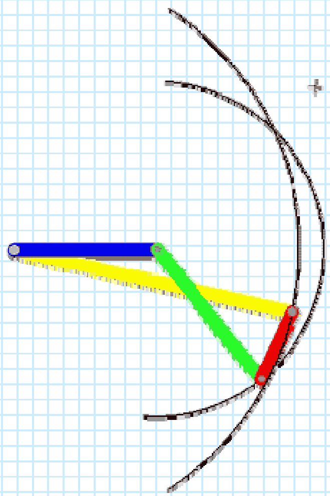
829 + 216 = 1045 > 485+415 = 900

Grashof's rule does not depend on how the links with different size are connected to each other.



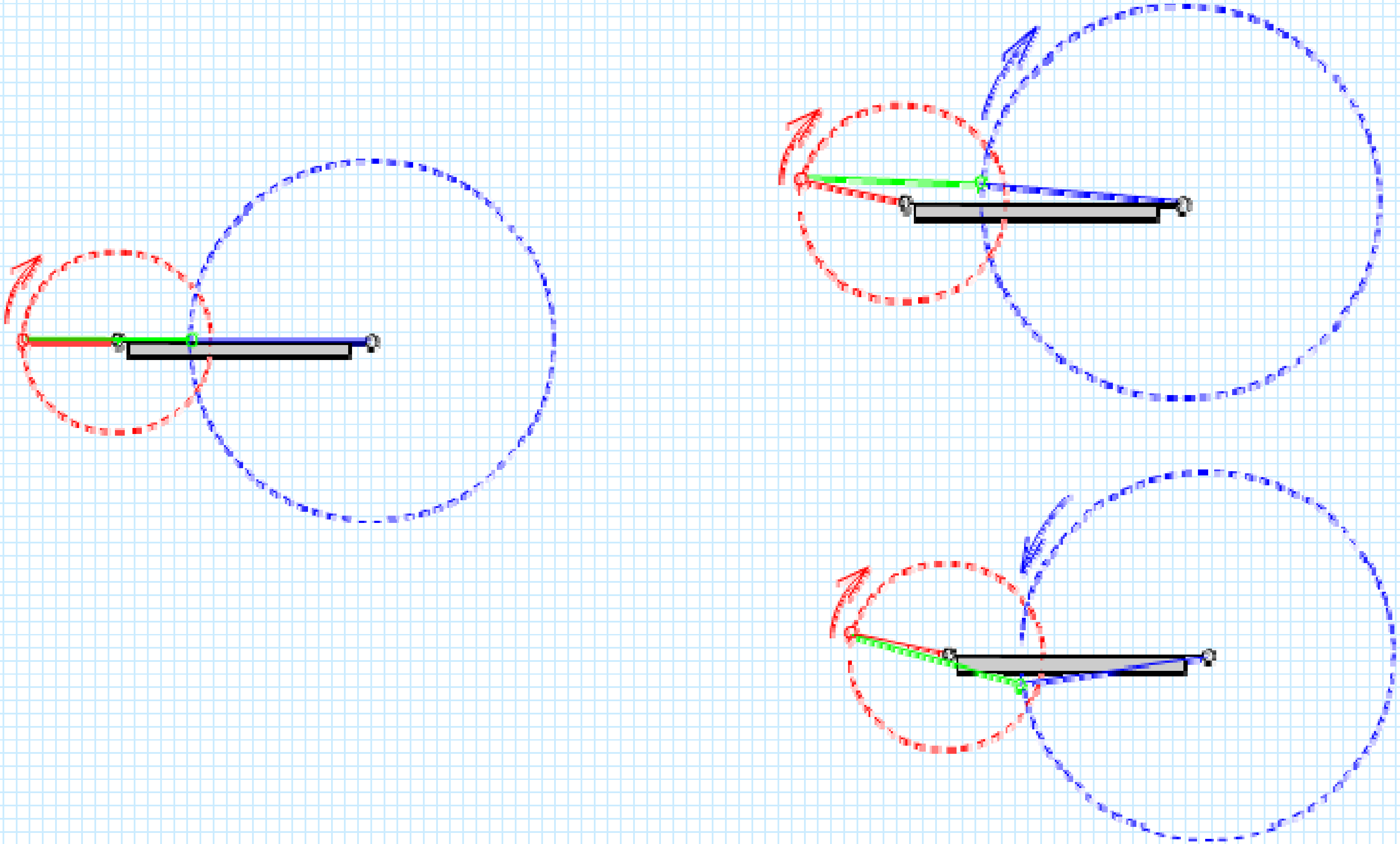


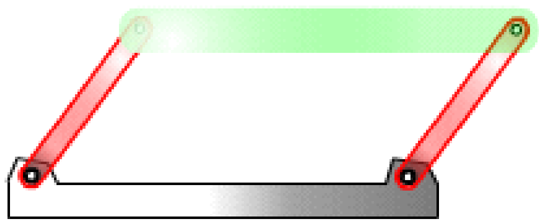




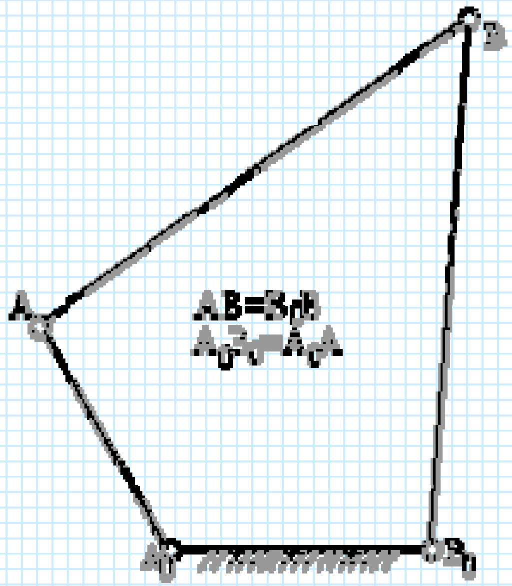
l + s = p + q

Same as *l* + s < p + q but "Change point" exists.
A position of the mechanism where all the joints are colinear (lie on a straight line)



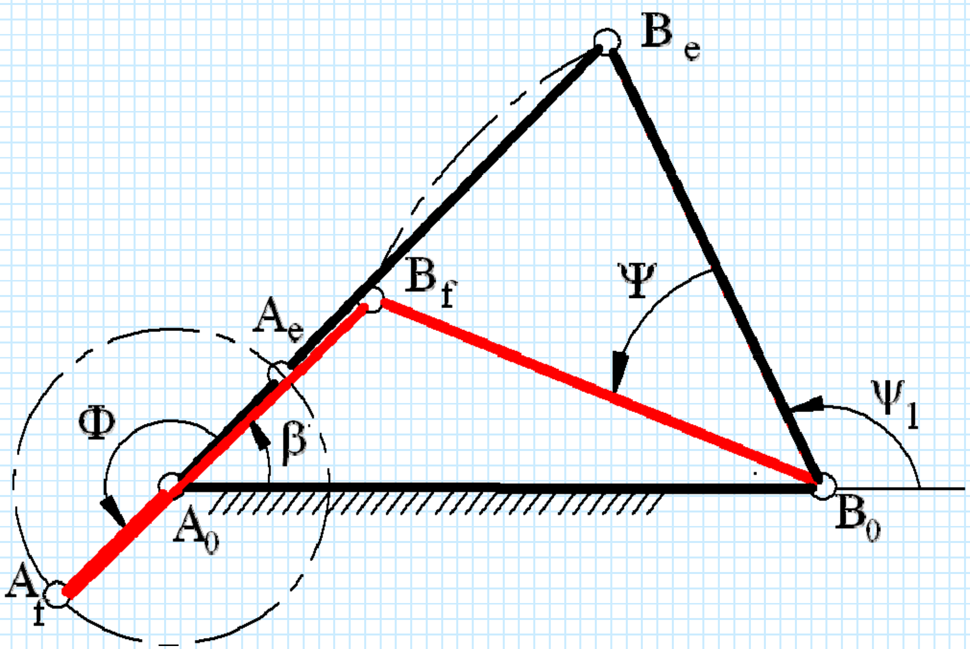


Paralelogram Linkage



Deltoid Linkage

Dead-Center Positions of Crank-Rocker Mechanisms

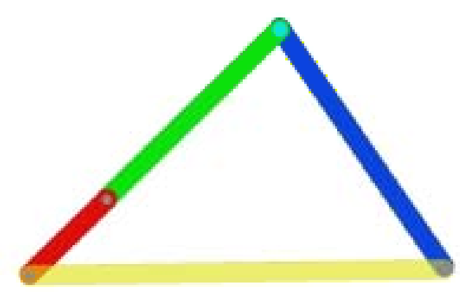


ψ= swing angle

ϕ= corresponding crank rotation

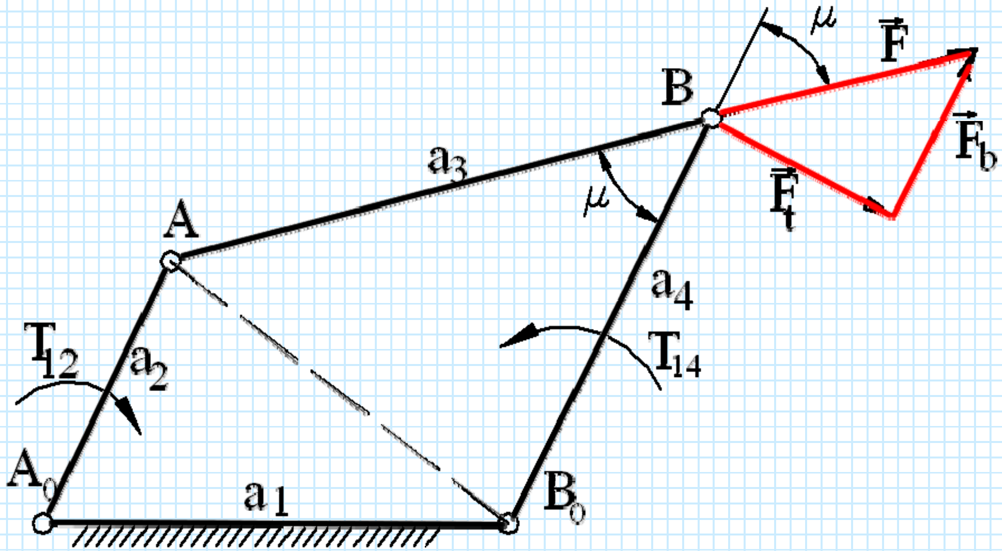
β= initial crank angle

Determine ψ and ϕ using cosine theorem.



$$TR = \frac{\text{time it takes for forward stroke}}{\text{time it takes for reverse stroke}} = \frac{\phi}{360^0 - \phi}$$

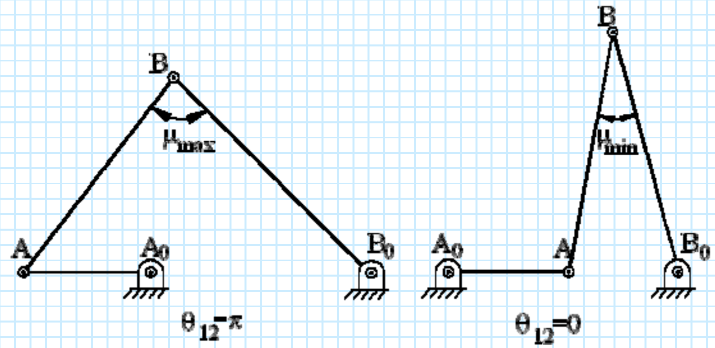
Transmission Angle



Transmission angle is a kinematic quantity which gives us an idea on how well the force is transmitted

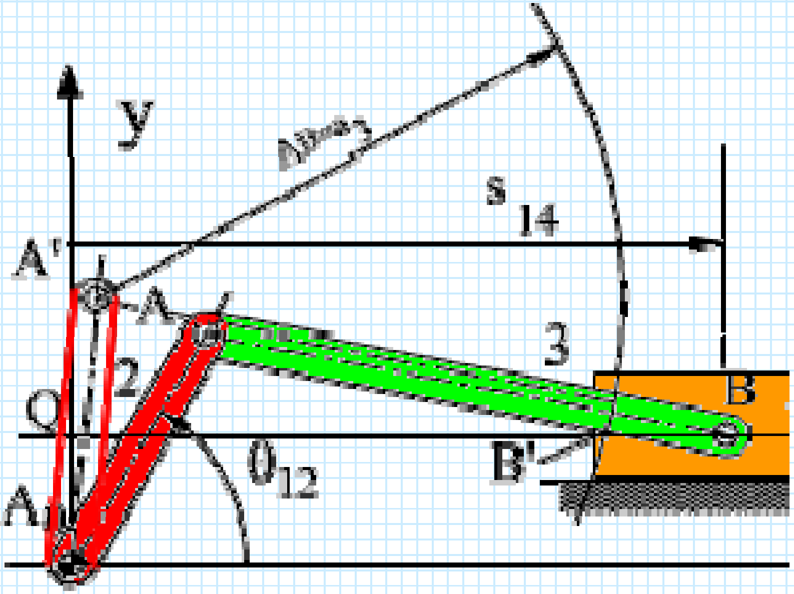
$$\cos\mu = \frac{a_4^2 + a_3^2 - a_1^2 - a_2^2}{2a_3a_4} + \frac{a_1a_2}{a_3a_4}\cos\theta_{12}$$

$$\cos\mu_{\substack{min\\max}} = \frac{a_4^2 + a_3^2 - a_1^2 - a_2^2}{2a_3a_4} \pm \frac{a_1a_2}{a_3a_4}$$



Examples:

Slider-Crank Mechanism

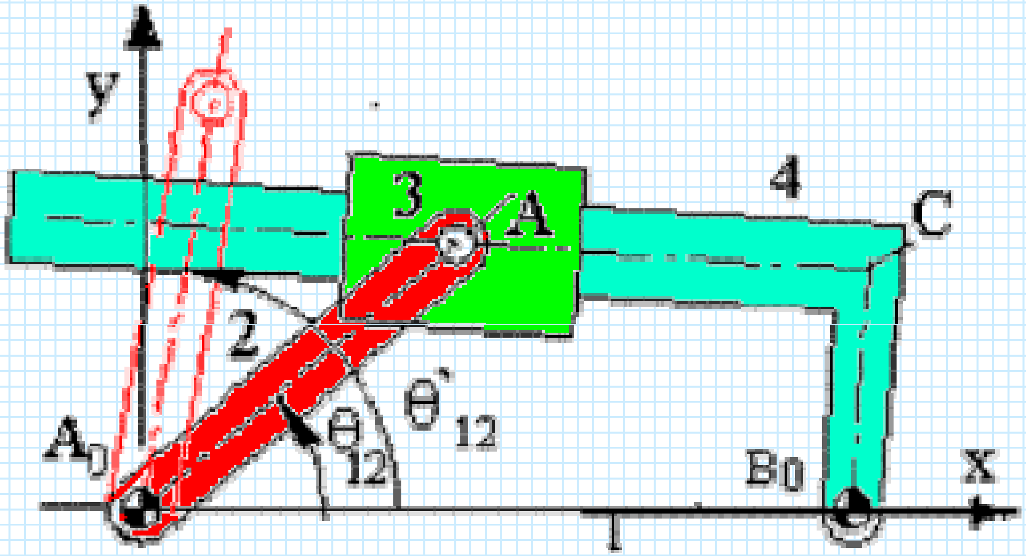


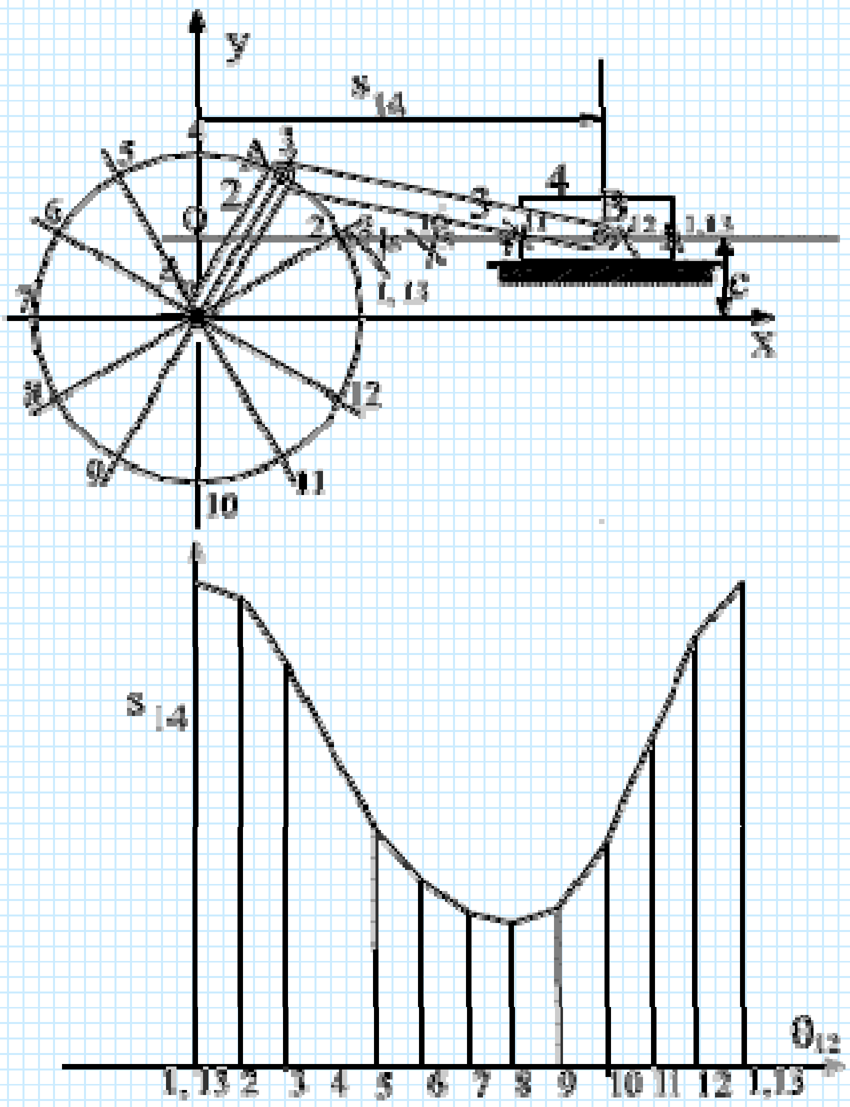
For the full rotatability of the crank:

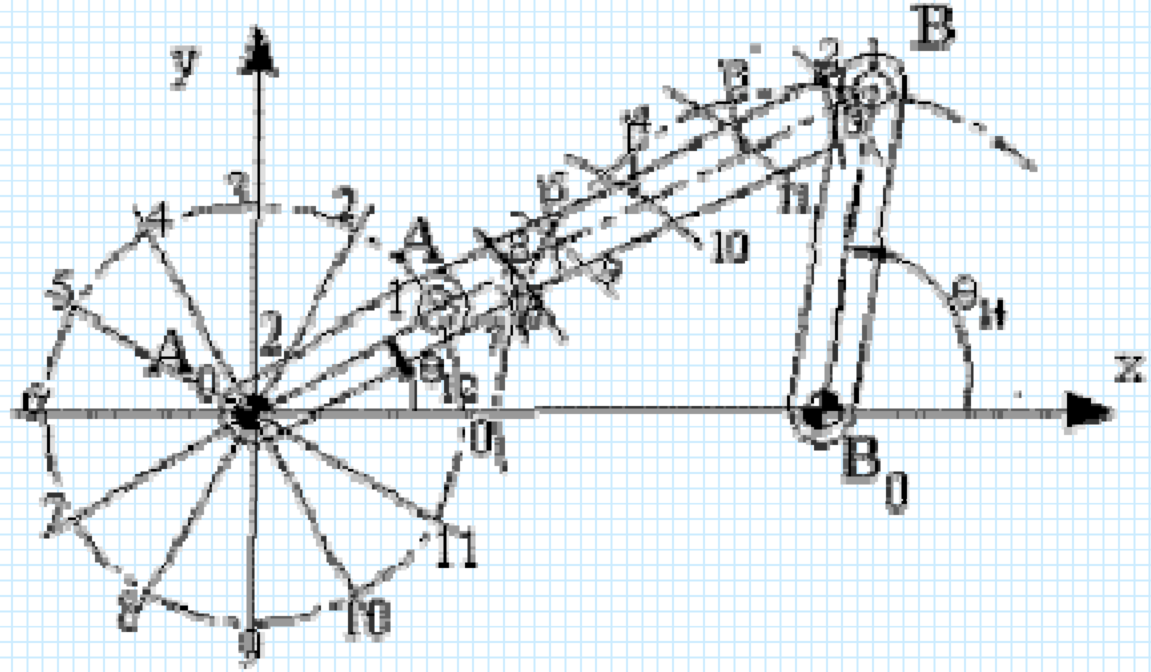
Eccentricity < $(a_3 - a_2)$ and

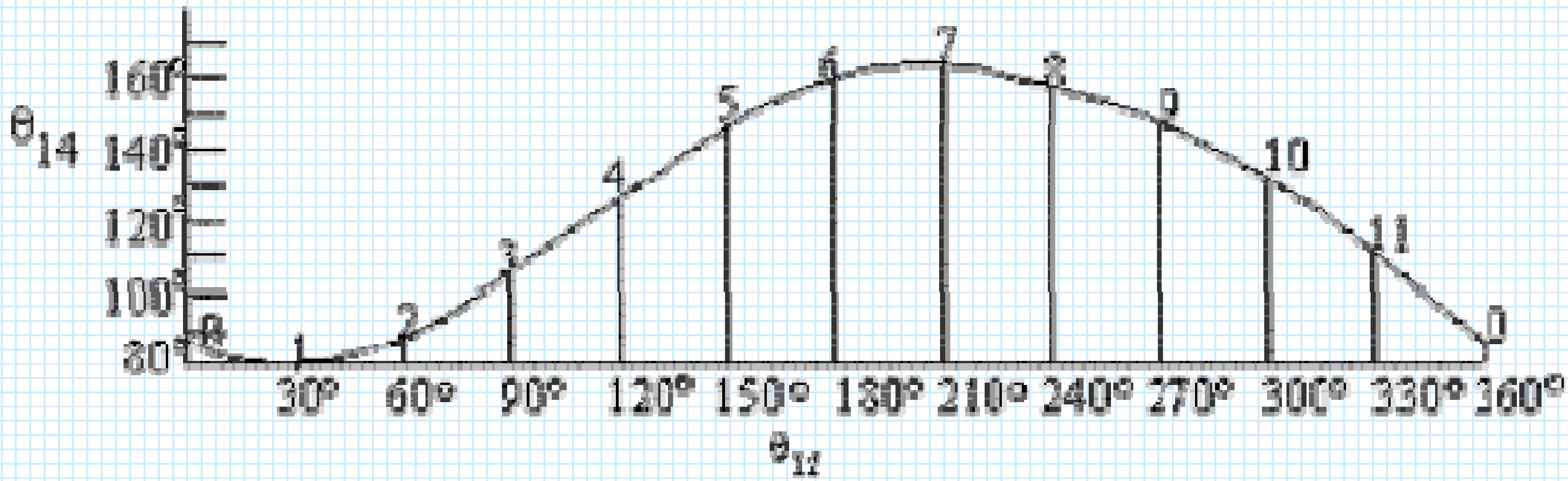
$a_3 > a_2$

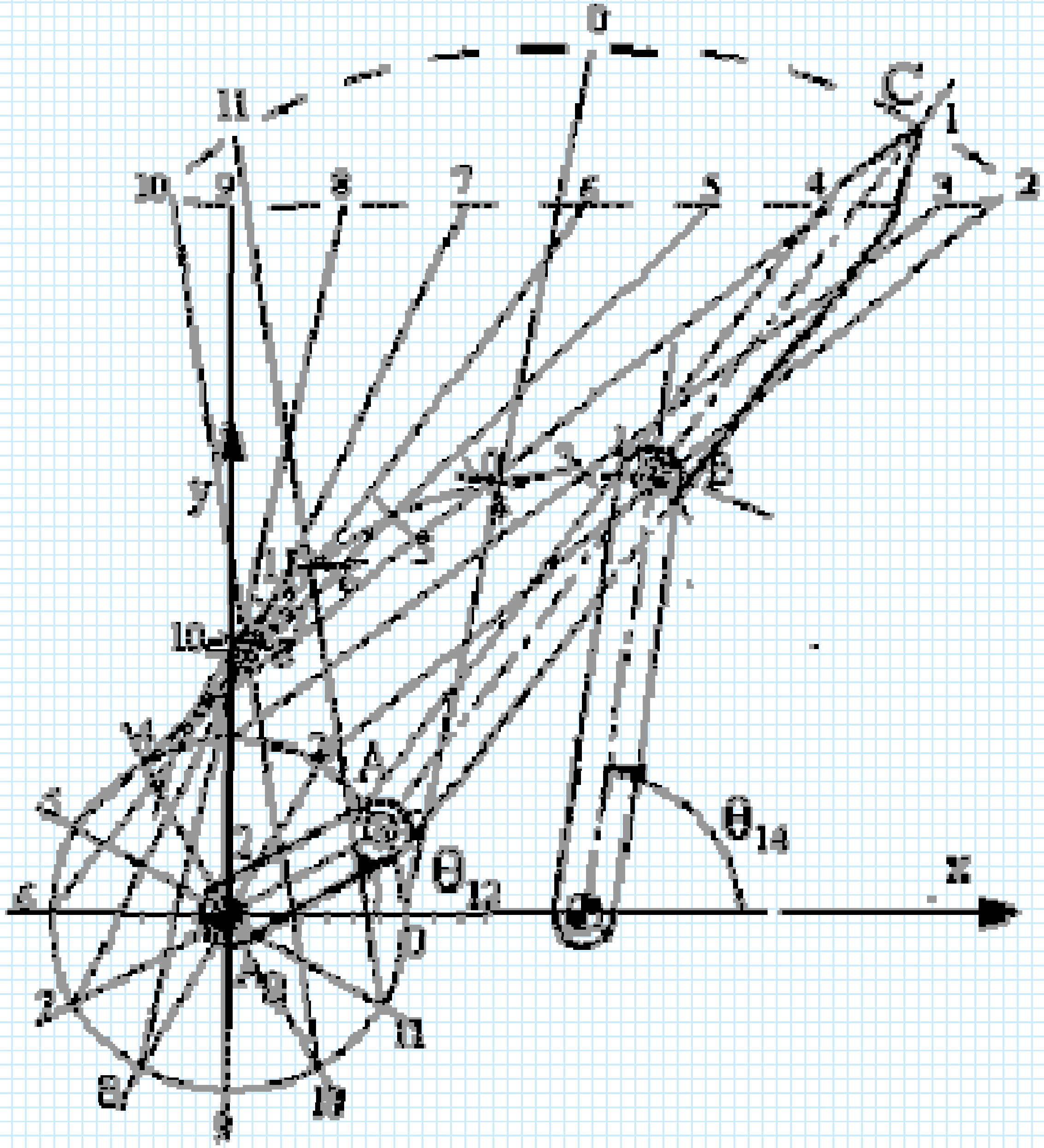
Inverted Slider-Crank Mechanism

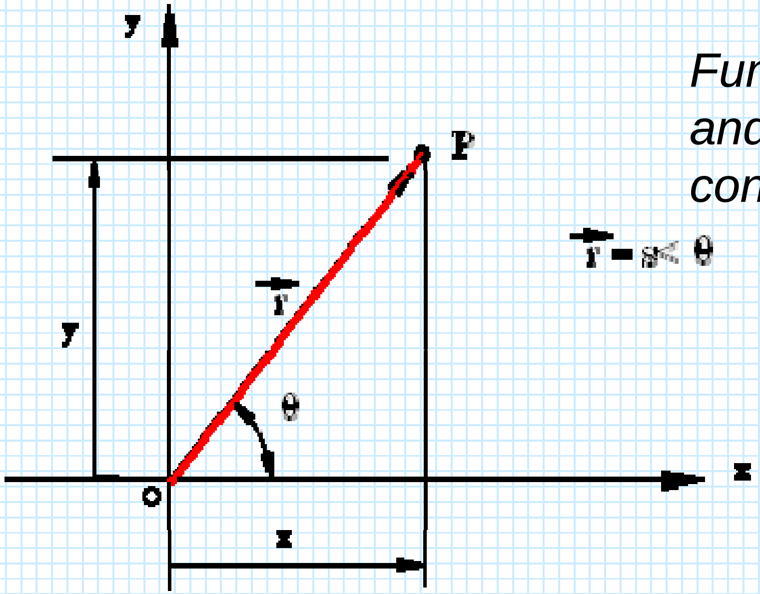



y
s_{14}
A
2
3
4
O
c
x
s_{14}
θ_{12}
1,13 2 3 4 5 6 7 8 9 10 11 12 1,13


y
x
A
A0
B
B0
θ14
θ12
1
2
3
4
5
6
7
8
9
10
11


θ14
160°
140°
120°
100°
80°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
360°
θ12
0
1
2
3
4
5
6
7
8
9
10
11

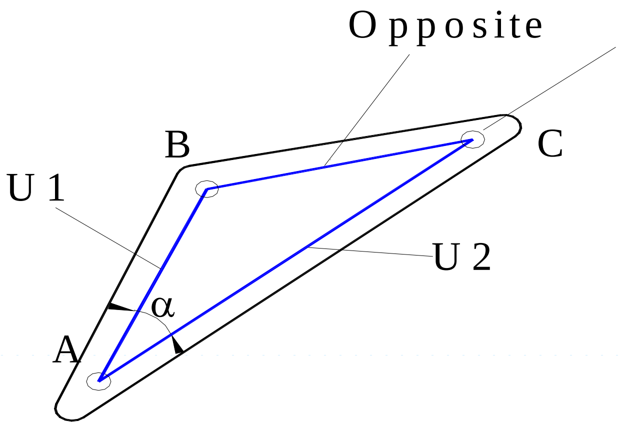

C
y
x
A
B
θ_{12}
θ_{14}

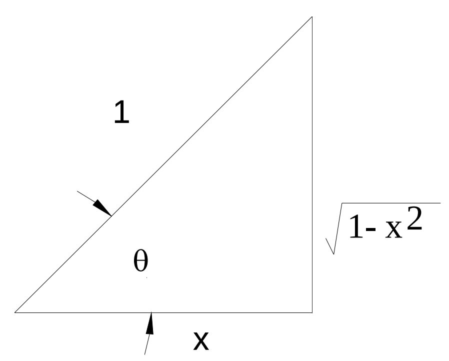


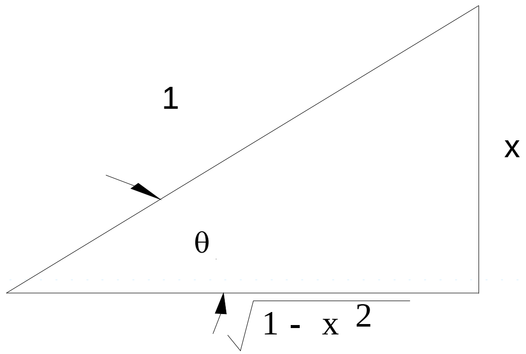
Function routines to determine the magnitude and the angle of a vector (rectangular to polar conversion)

```
Function Mag(X, Y)
‘ returns the magnitude of the vector
Mag = Sqr(X ^ 2 + Y ^ 2)
End Function

Function Ang(X, Y)
‘ returns the angle the vector makes wr to +ve x axis
Dim AA, Pi
Pi=4*Atn(1)
    If Abs(X) > eps Then
        AA = Atn(Y / X)
        If X < 0 Then
            AA = AA + Pi
            Else:
            If Y < 0 Then AA = AA + 2*Pi
        End If
        Else:
        If Y > 0 Then AA = Pi/2  Else AA = -Pi / 2
     End If
    Ang = AA
End Function
```







Function routines to solve an unknown angle or length of a triangle using cosine theorem

```
Function AngCos(u1, u2, Opposite)
'returns the angle alfa
   Dim U
   U = (u1 * u1 + u2 * u2 - Opposite * Opposite) / (2 * u1 * u2)
   AA = Acos(U)
   AngCos = AA
End Function

Function MagCos(u1, u2, Angle)
' returns the length of the side opposite to the side
MagCos = Sqr(u1 * u1 + u2 * u2 - 2 * u1 * u2 * Cos(Angle))
End Function

Function Acos(X)
   Acos = Atn(-X / Sqr(-X * X + 1)) + 2 * Atn(1)
End Function

Function Asin(X)
Asin = Atn(X / Sqr(-X * X + 1))
End Function
```

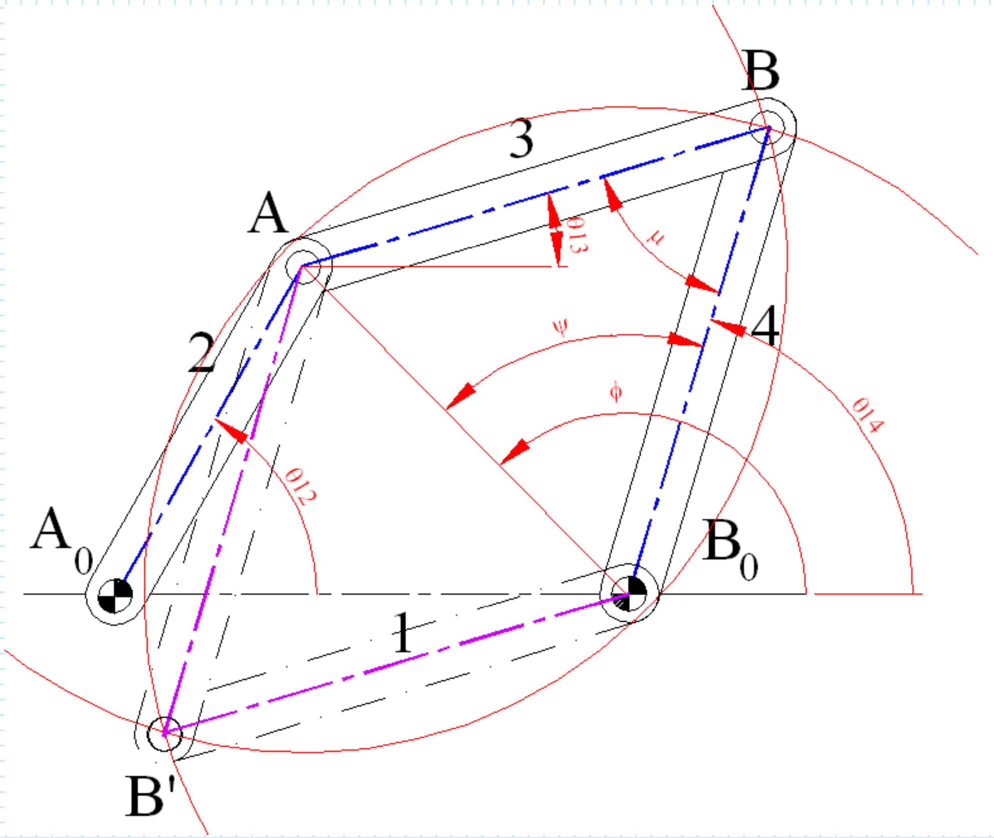
Stepwise Solution

Write a set of equations which can be solved in steps to yield a complete analysis of the mechanism

Or

Derive an algorithm to perform a complete position analysis

Example: Four-bar



$B_0A = s_x + is_y = s\angle\phi$

$s_x = a_2\cos(\theta_{12}) - a_1$ (1)

$s_y = a_2\sin(\theta_{12})$ (2)

$$s = \sqrt{s_x^{\,2} + s_y^{\,2}} \quad (3)$$

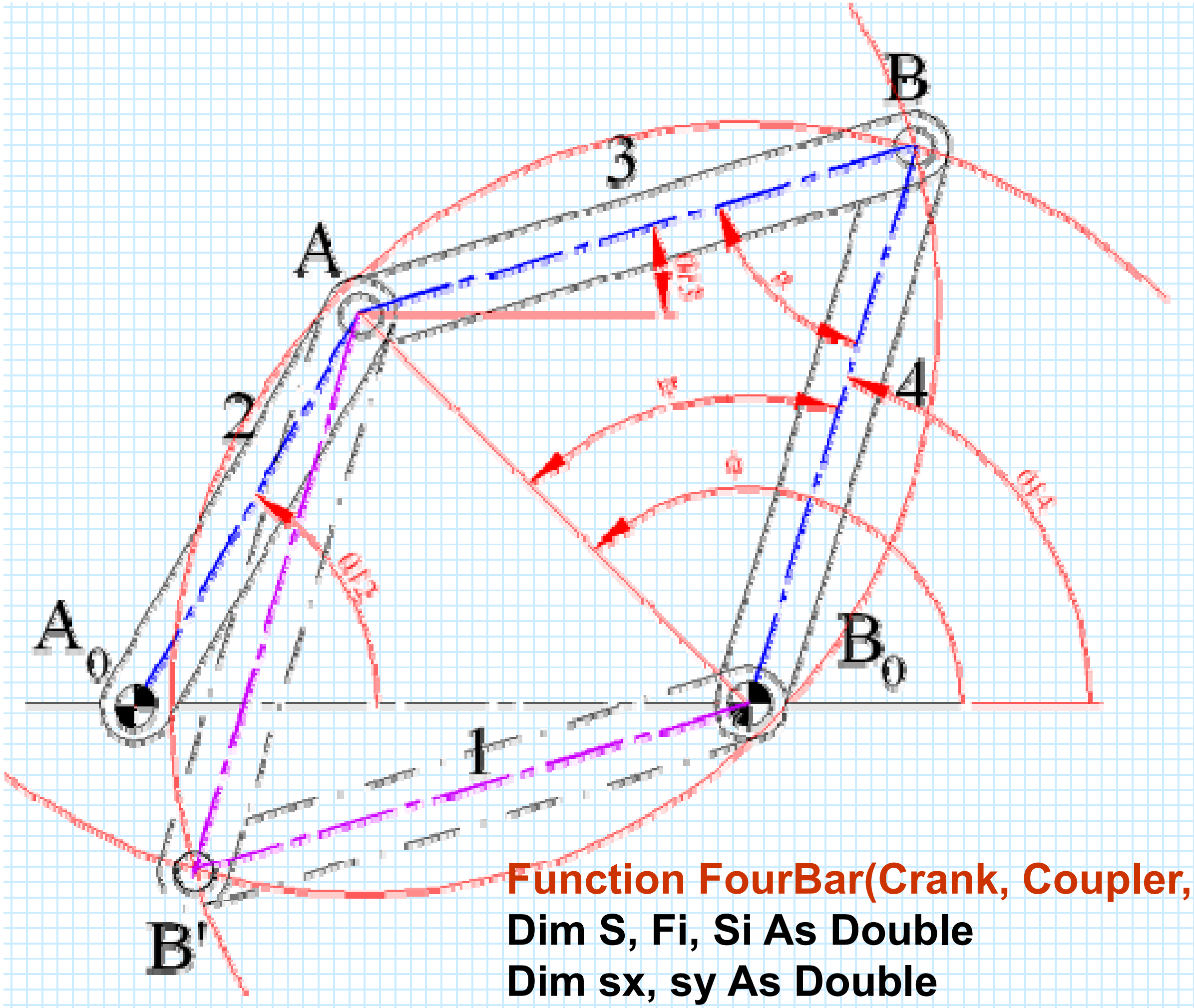
$$\phi = a\tan^{-1}(s_x, s_y) \quad (4)$$

$$\psi = \cos^{-1}\left[\left(a_4^2 + s^2 - a_3^2\right)/2a_4 s\right] \quad (5)$$

$$\mu = \pm\cos^{-1}\left[\left(a_3^2 + a_4^2 - s^2\right)/2a_3 a_4\right] \quad (6)$$

$\theta_{14} = \phi \pm \psi$ (7)

$\theta_{13} = \theta_{14} - \mu$ (8)



Given:

The link lengths a_1, a_2, a_3, a_4,

the configuration (config)

The input crank angle θ_{12}

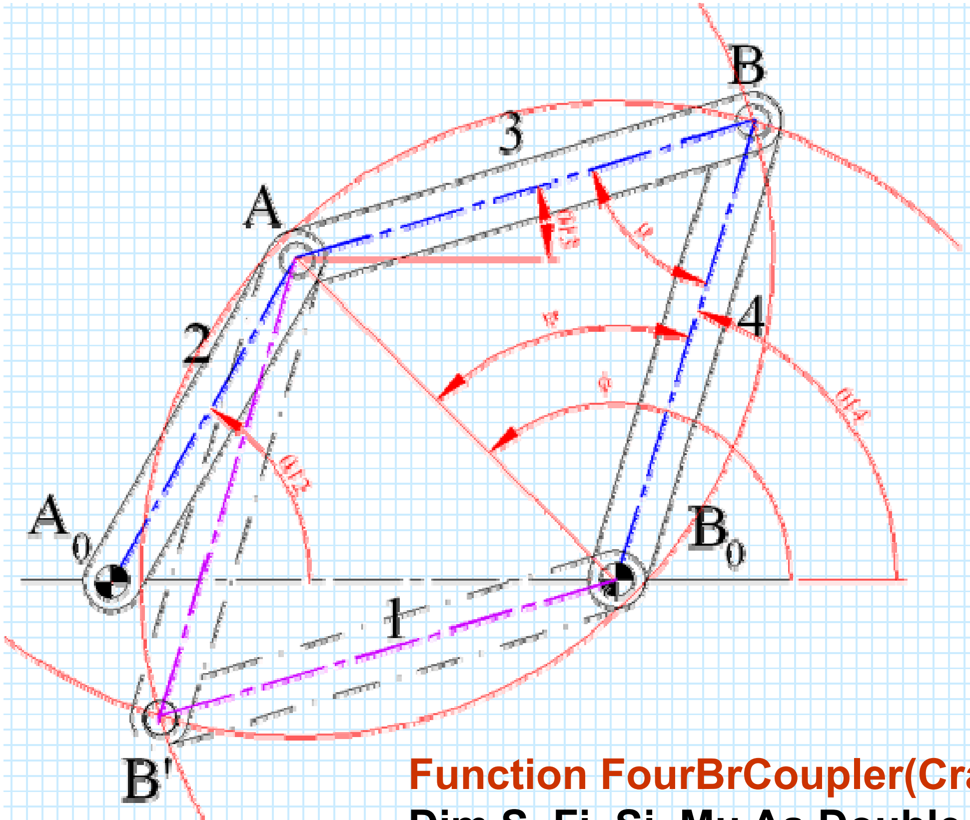
Determine: The position of the links

Solve equations 1-8

Config= +1 or -1(cross configuration)

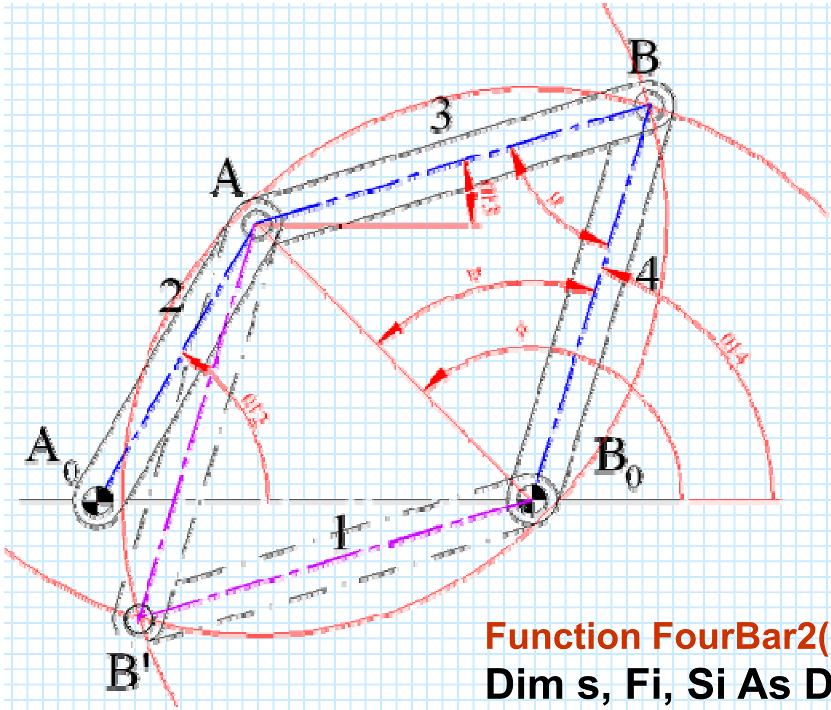
This function routine returns the value of θ_{14}

```
Function FourBar(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim S, Fi, Si As Double
Dim sx, sy As Double
sx= -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
S = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
FourBar = Fi - Config * Si
End Function
```



This function routine returns the value of θ_{13}

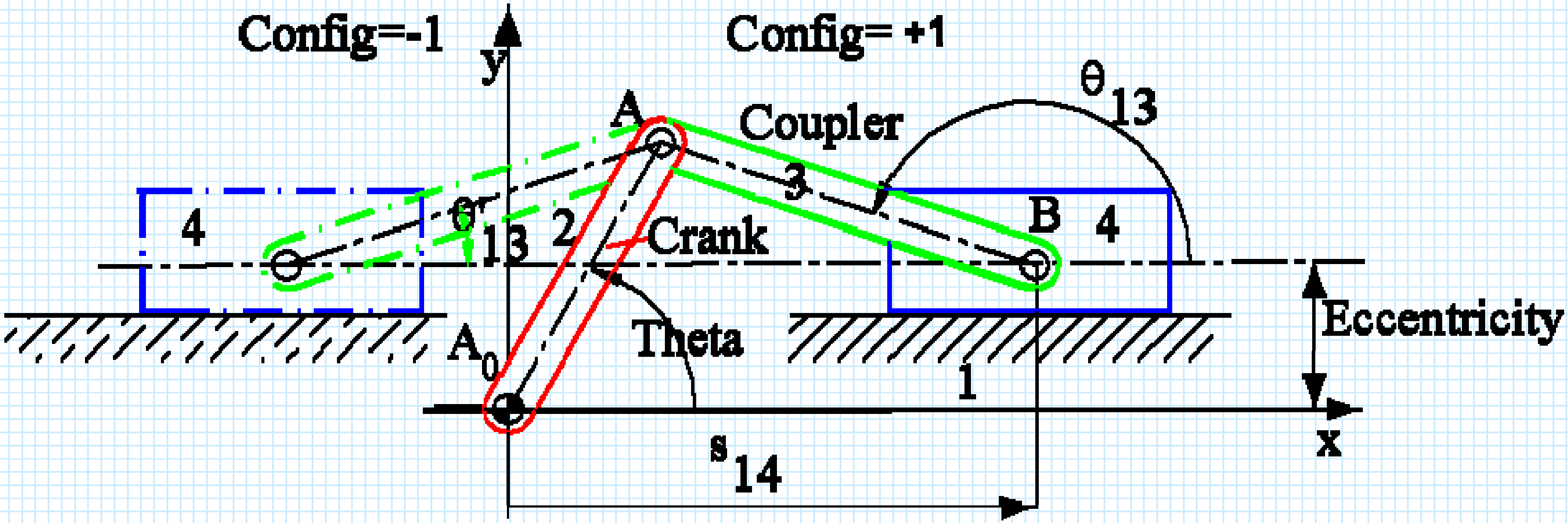
```
Function FourBrCoupler(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim S, Fi, Si, Mu As Double
Dim sx, sy, Theta4 As Double
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
S = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Theta4= Fi - Config * Si
Mu = AngCos(Coupler, Rocker, S)
FourBrCoupler = Theta4 - Mu
End Function
```




```
Function FourBar2(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim s, Fi, Si As Double
Dim sx, sy As Double
Dim A(2)
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
s = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Mu = AngCos(Coupler, Rocker, s)
A(1) = Fi - Config * Si
A(0) = A(1) - Mu
FourBar2 = A
End Function
```

This function routine returns both values θ_{13} and θ_{14}

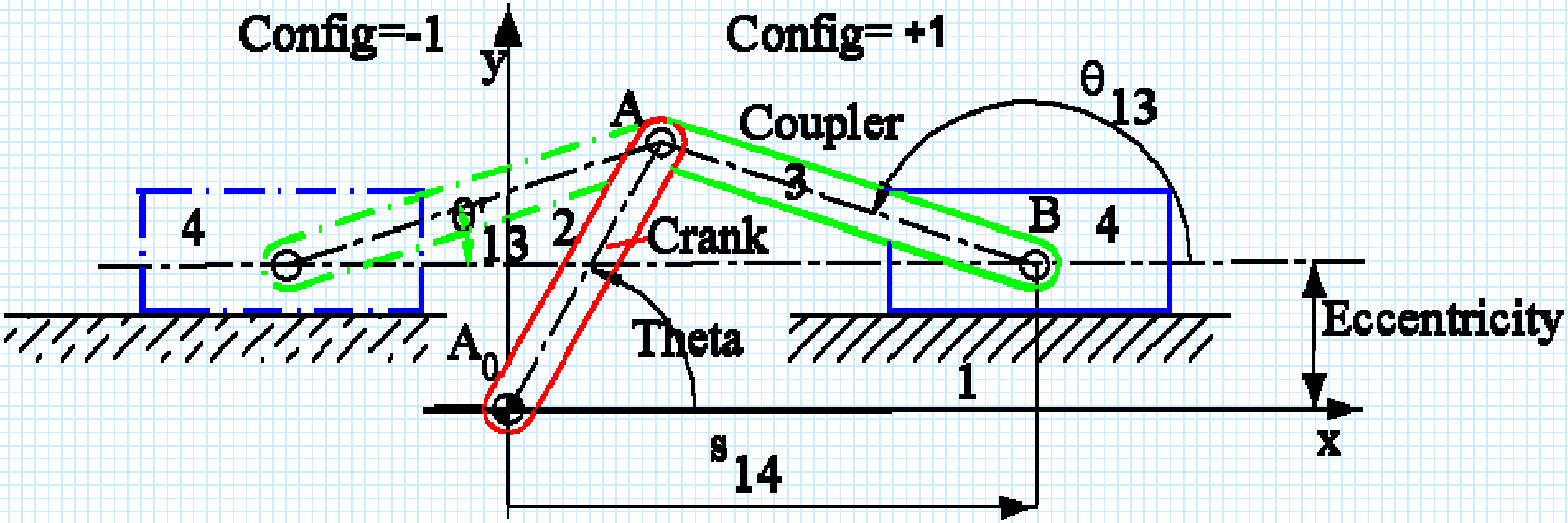
Example: Slider-Crank Mechanism



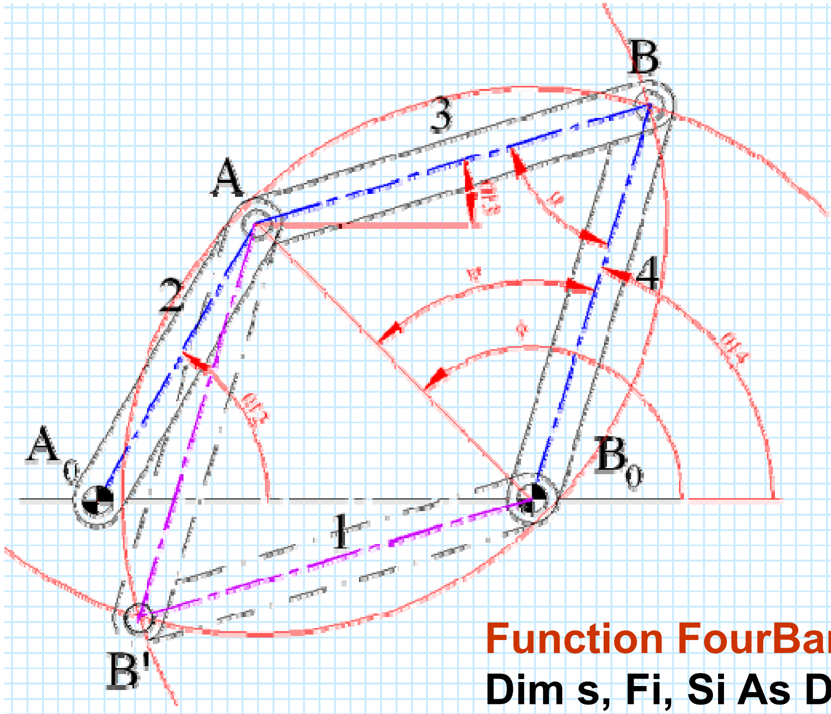
$$\phi = \sin^{-1}\left[\frac{a_2 \sin\theta_{12} - a_1}{a_3}\right]$$

If Config= +1 then $\theta_{13}=\pi-\phi$
If Config= -1 then $\theta_{13}=\phi$
$s_{14}=a_2\cos\theta_{12}-a_3\cos\theta_{13}$

Example: Slider-Crank mechanism



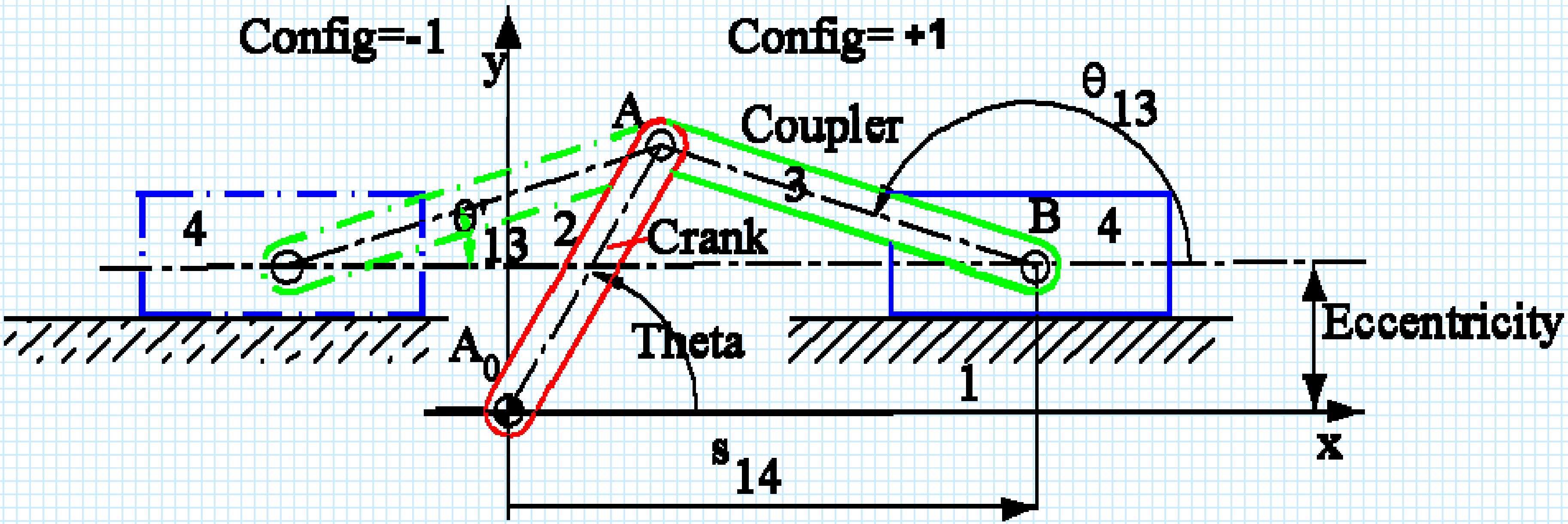

```
Function SliderCrank(Crank, Coupler, Eccentricity, Config, Theta)
Dim Fi As Double
Fi = Asin((Crank * Sin(Theta) - Eccentricity) / Coupler)
If Config = 1 Then Fi = 4 * Atn(1) - Fi
SliderCrank = Crank * Cos(Theta) - Coupler * Cos(Fi)
End Function
```



Function with double argument

```
Function FourBar2(Crank, Coupler, Rocker, Fixed, Config, Theta)
Dim s, Fi, Si As Double
Dim sx, sy As Double
Dim A(2)
sx = -Fixed + Crank * Cos(Theta)
sy = Crank * Sin(Theta)
s = Mag(sx, sy)
Fi = Ang(sx, sy)
Si = AngCos(Rocker, S, Coupler)
Mu = AngCos(Coupler, Rocker, s)
A(1) = Fi - Config * Si
A(0) = A(1) - Mu
FourBar2 = A
End Function
```

This function routine returns both values θ_{13} and θ_{14}




```
Function SliderCrank2(Crank, Coupler, Eccentricity, Config, Theta)
Dim Fi As Double
Dim A(2) As Double
Fi = Asin((Crank * Sin(Theta) - Eccentricity) / Coupler)
If Config = 1 Then Fi = 4 * Atn(1) - Fi
A(0) = Fi
A(1) = Crank * Cos(Theta) - Coupler * Cos(Fi)
SliderCrnk2 = A
End Function
```